STAT/MA 41600
Practice Problems: December 5, 2014
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1. a. For $0 \leq x_1 \leq 20$, we have $f_{X_{(1)}}(x_1) = \binom{4}{0,1,3} \left(\frac{1}{20}\right) \left(\frac{x_1}{20}\right)^0 \left(1 - \frac{x_1}{20}\right)^3 = (4) \left(\frac{1}{20}\right)^4 (20 - x_1)^3$. Otherwise, $f_{X_{(1)}}(x_1) = 0$.

b. For $0 \leq x_2 \leq 20$, we have $f_{X_{(2)}}(x_2) = \binom{4}{1,1,2} \left(\frac{1}{20}\right) \left(\frac{x_2}{20}\right)^1 \left(1 - \frac{x_2}{20}\right)^2 = (12) \left(\frac{1}{20}\right)^4 (x_2)(20 - x_2)^2$. Otherwise, $f_{X_{(2)}}(x_2) = 0$.

2. a. We integrate using **1a** and get $\mathbb{E}(X_{(1)}) = \int_0^{20} (x_1)(4) \left(\frac{1}{20}\right)^4 (20 - x_1)^3 \, dx_1 = 4$.

b. We integrate using **1b** and get $\mathbb{E}(X_{(2)}) = \int_0^{20} (x_2)(12) \left(\frac{1}{20}\right)^4 (x_2)(20 - x_2)^2 \, dx_2 = 8$.

3. a. For $0 < x_1$, we have $f_{X_{(1)}}(x_1) = \binom{2}{0,1,1} \frac{1}{10} e^{-x_1/10} \left(1 - e^{-x_1/10}\right)^0 \left(e^{-x_1/10}\right)^1 = \frac{2}{10} (e^{-x_1/10})^2$. Otherwise, $f_{X_{(1)}}(x_1) = 0$.

b. For $0 < x_2$,
we have $f_{X_{(2)}}(x_2) = \binom{2}{1,1,0} \frac{1}{10} e^{-x_2/10} \left(1 - e^{-x_2/10}\right)^1 \left(e^{-x_2/10}\right)^0 = \frac{2}{10} (e^{-x_2/10})(1 - e^{-x_2/10})$. Otherwise, $f_{X_{(2)}}(x_2) = 0$.

4. a. We integrate using **3a** and get $\mathbb{E}(X_{(1)}) = \int_0^{\infty} (x_1)(\frac{2}{10})(e^{-x_1/10})^2 \, dx_1 = 5$.

b. We integrate using **3b** and get $\mathbb{E}(X_{(2)}) = \int_0^{\infty} (x_2)(\frac{2}{10})(e^{-x_2/10})(1 - e^{-x_2/10}) \, dx_2 = 15$.

5. a. The CDF of each of the random variables is, for $0 < a < 1$,

$$F_X(a) = P(X \leq a) = \int_0^a 6(x - x^2) \, dx = -2a^3 + 3a^2.$$

Thus for $0 < x_1 < 1$,

$$\begin{aligned} f_{X_{(1)}}(x_1) &= \binom{2}{0,1,1} f_X(x_1)(1 - F_X(x_1)) \\ &= \binom{2}{0,1,1} \left(6(x_1 - x_1^2)\right) \left(1 + 2x_1^3 - 3x_1^2\right) \\ &= 12x_1 - 12x_1^2 - 36x_1^3 + 60x_1^4 - 24x_1^5 \end{aligned}$$

Otherwise, $f_{X_{(1)}}(x_1) = 0$.

b. We integrate using **5a** and get $\mathbb{E}(X_{(1)}) = \int_0^1 (x_1)(12x_1 - 12x_1^2 - 36x_1^3 + 60x_1^4 - 24x_1^5) dx_1 = 13/35$.