STAT/MA 41600
Practice Problems: November 12, 2014
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We always use Z to denote a standard normal random variable in these answers.

1a. We have $\mathbb{E}(Y) = \mathbb{E}(\frac{X_1+X_2+X_3+X_4+X_5}{5}) = \frac{1}{5}(\mathbb{E}(X_1)+\cdots+\mathbb{E}(X_5)) = \frac{1}{5}(8.2+\cdots+8.2) = 8.2$. The X_j's are independent, so $\mathrm{Var}(Y) = \mathrm{Var}(\frac{X_1+X_2+X_3+X_4+X_5}{5}) = \frac{1}{25}(\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_5)) = \frac{1}{25}(32.49+\cdots+32.49) = \frac{32.49}{5} = 6.498$.

1b. We have $\mathbb{E}(Y) = \mathbb{E}(\frac{X_1+X_2+\cdots+X_n}{n}) = \frac{1}{n}(\mathbb{E}(X_1)+\cdots+\mathbb{E}(X_n)) = \frac{1}{n}(\mu+\cdots+\mu) = \mu$. The X_j's are independent, so $\mathrm{Var}(Y) = \mathrm{Var}(\frac{X_1+X_2+\cdots+X_n}{n}) = \frac{1}{n^2}(\mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_n)) = \frac{1}{n^2}(\sigma^2+\cdots+\sigma^2) = \frac{\sigma^2}{n}$.

2. Let Y_1, Y_2, Y_3 be the amounts in the three people's accounts. So $X = Y_1+Y_2+Y_3$. So X is the sum of independent normals, and thus X is normal too, with $\mathbb{E}(X) = \mathbb{E}(Y_1+Y_2+Y_3) = \mathbb{E}(Y_1)+\mathbb{E}(Y_2)+\mathbb{E}(Y_3) = 1325+1325+1325 = 3975$, and $\mathrm{Var}\,X = \mathrm{Var}(Y_1+Y_2+Y_3) = \mathrm{Var}(Y_1)+\mathrm{Var}(Y_2)+\mathrm{Var}(Y_3) = 25^2+25^2+25^2 = 1875$, so $\sigma_X = 43.30$. Thus $P(X > 4000) = P(\frac{X-3975}{43.30} > \frac{4000-3975}{43.30}) = P(Z > .58) = 1 - P(Z \leq .58) = 1 - .7190 = .2810$.

3. Let X_1, X_2, X_3, X_4 be the lengths of time for the four people's haircuts. So $Y = X_1 + X_2 + X_3 + X_4$ is the total length of time. So Y is the sum of independent normals, and thus Y is normal too, with $\mathbb{E}(Y) = \mathbb{E}(X_1+X_2+X_3+X_4) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\mathbb{E}(X_3)+\mathbb{E}(X_4) = 23.8+23.8+23.8+23.8 = 95.2$, and $\mathrm{Var}\,Y = \mathrm{Var}(X_1+X_2+X_3+X_4) = \mathrm{Var}(X_1)+\mathrm{Var}(X_2)+\mathrm{Var}(X_3)+\mathrm{Var}(X_4) = 5^2+5^2+5^2+5^2 = 100$, so $\sigma_Y = 10$. Thus $P(Y \leq 90) = P(\frac{Y-95.2}{10} \leq \frac{90-95.2}{10}) = P(Z \leq -.52) = P(Z \geq .52) = 1 - P(Z < .52) = 1 - .6985 = .3015$.

4. As in problem 1b above, $\mathbb{E}(Y) = 64$, and $\mathrm{Var}(Y) = \frac{12.8^2}{10} = 16.384$, so $P(Y > 60) = P(\frac{Y-64}{\sqrt{16.384}} > \frac{60-64}{\sqrt{16.384}}) = P(Z > -0.99) = P(Z < 0.99) = .8389$.

5. Let $Y = X_1+\cdots+X_7$ be the total quantity of sugar, where X_j is the amount of sugar in the jth piece. So Y is the sum of independent normals, and thus Y is normal too, with $\mathbb{E}(Y) = \mathbb{E}(X_1+\cdots+X_7) = \mathbb{E}(X_1)+\cdots+\mathbb{E}(X_7) = 22+\cdots+22 = 154$, and $\mathrm{Var}\,Y = \mathrm{Var}(X_1+\cdots+X_7) = \mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_7) = 8+\cdots+8 = 56$, so $\sigma_Y = 7.48$. Thus $P(Y \geq 150) = P(\frac{Y-154}{7.48} \geq \frac{150-154}{7.48}) = P(Z \geq -.53) = P(Z \leq .53) = .7019$.