STAT/MA 41600
Practice Problems: October 27, 2014
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1.

a. *Method #1:* Since we saw $\mathbb{E}(X) = 1$ and $\mathbb{E}(Y) = 1$ in the previous problem set, then $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 1 + 1 = 2$.

Method #2: We have $\mathbb{E}(X+Y) = \int_0^3 \int_0^{3-x} (x+y)(\frac{2}{9})\,dy\,dx = \int_0^3 (xy + \frac{y^2}{2})\left(\frac{2}{9}\right)\Big|_{y=0}^{3-x} dx = \int_0^3 (1 - \frac{x^2}{9})\,dx = (x - \frac{x^2}{9})\Big|_{x=0}^{3} = (3 - \frac{3^2}{9}) = 2$.

b. *Method #1:* Since we know from the previous problem set that $f_X(x) = \frac{2}{9}(3-x)$ for $0 \le x \le 3$, then we can integrate $\mathbb{E}(X^2) = \int_0^3 x^2(\frac{2}{9})(3-x)\,dx = \int_0^3 (\frac{2}{9})(3x^2 - x^3)\,dx = (\frac{2}{9})(x^3 - \frac{x^4}{4})\Big|_{x=0}^{3} = (\frac{2}{9})(27 - \frac{81}{4}) = 3/2$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/2 - 1^2 = 1/2$.

Method #2: We can integrate $\mathbb{E}(X^2) = \int_0^3 \int_0^{3-x} (x^2)(\frac{2}{9})\,dy\,dx = \int_0^3 (3x^2 - x^3)(\frac{2}{9})\,dx = \frac{2}{9}(x^3 - \frac{x^4}{4})\Big|_{x=0}^{3} = \frac{2}{9}(27 - \frac{81}{4}) = 3/2$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3/2 - 1^2 = 1/2$.

2.

a. *Method #1:* Since we know from the previous problem set that $f_X(x) = \frac{1}{6}(4-x)$ for $0 \le x \le 2$, then we can integrate $\mathbb{E}(X^2) = \int_0^2 x^2(\frac{1}{6})(4-x)\,dx = \int_0^2 (\frac{1}{6})(4x^2 - x^3)\,dx = (\frac{1}{6})(\frac{4x^3}{3} - \frac{x^4}{4})\Big|_{x=0}^{2} = (\frac{1}{6})(\frac{32}{3} - 4) = 10/9$. We saw $\mathbb{E}(X) = 8/9$ in the previous problem set. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10/9 - (8/9)^2 = 26/81$.

Method #2: We have $\mathbb{E}(X^2) = \int_0^2 \int_0^{12-3x} (x^2)(\frac{1}{18})\,dy\,dx = \int_0^2 (\frac{1}{18})(12x^2 - 3x^3)\,dx = (\frac{1}{18})(4x^3 - \frac{3x^4}{4})\Big|_{x=0}^{2} = 10/9$. So $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10/9 - (8/9)^2 = 26/81$.

b. *Method #1:* Since we know from the previous problem set that $f_Y(y) = 1/6$ for $0 \le y \le 6$ and $f_Y(y) = \frac{12-y}{54}$ for $6 \le x \le 12$, then we can integrate $\mathbb{E}(Y^2) = \int_0^6 y^2(\frac{1}{9})\,dy + \int_6^{12} y^2(\frac{12-y}{54})\,dy = \frac{6^3}{3}(\frac{1}{9}) + (\frac{12y^3/3 - y^4/4}{54})\Big|_{y=6}^{12} = 30$. We saw $\mathbb{E}(Y) = 14/3$ in the previous problem set. So $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 30 - (14/3)^2 = 74/9$.

Method #2: We have $\mathbb{E}(Y^2) = \int_0^2 \int_0^{12-3x} (y^2)(\frac{1}{18})\,dy\,dx = \int_0^2 (\frac{1}{18})\frac{(12-3x)^3}{3}\,dx$. Using $u = 12 - 3x$, $du = -3dx$, we get $\mathbb{E}(Y^2) = \int_6^{12} (\frac{1}{18})\frac{u^3}{9}\,du = \frac{1}{18}(\frac{12^4}{36} - \frac{6^4}{36}) = 30$. So $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 30 - (14/3)^2 = 74/9$.

3. Since X and Y are independent, then $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$. We already saw in the previous problem set that $f_X(x) = 2e^{-2x}$ for $x > 0$, and $f_X(x) = 0$ otherwise; also $\mathbb{E}(X) = 1/2$. We already saw $f_Y(x) = 7e^{-7y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise; also $\mathbb{E}(Y) = 1/7$.

Now we compute $\mathbb{E}(X^2) = \int_0^\infty x^2 2e^{-2x}\,dx$, and we use $u = x^2$ and $dv = 2e^{-2x}\,dx$, so $du = 2x\,dx$ and $v = -e^{-2x}$, to get $\mathbb{E}(X^2) = -x^2e^{-2x}|_{x=0}^\infty - \int_0^\infty -2xe^{-2x}\,dx = \int_0^\infty x\,2e^{-2x}\,dx$. We can either integrate a second time, by parts, or just recognize that the integral here is equal to $\mathbb{E}(X)$, which we already calculated in the previous problem set, question #3. So altogether we have $\mathbb{E}(X^2) = \mathbb{E}(X) = 1/2$. Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1/2 - (1/2)^2 = 1/4$.

Similarly $\mathbb{E}(Y^2) = \int_0^\infty y^2 7e^{-7y}\,dy$, and we use $u = y^2$ and $dv = 7e^{-7y}\,dy$, so $du = 2y\,dy$ and $v = -e^{-7y}$, to get $\mathbb{E}(Y^2) = -y^2e^{-7y}|_{y=0}^\infty - \int_0^\infty -2ye^{-7y}\,dy = \frac{2}{7}\int_0^\infty y\,7e^{-7y}\,dy$. We can either integrate a second time, by parts, or just recognize that the integral here is equal to $\mathbb{E}(Y)$, which we already calculated in the previous problem set, #3. So altogether $\mathbb{E}(Y^2) = \frac{2}{7}\mathbb{E}(Y) = (2/7)(1/7) = 2/49$. Thus $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/49 - (1/7)^2 = 1/49$.

4. *Method #1:* We know $\mathbb{E}(Y) = 1/9$ from the previous problem set, and $f_Y(y) = 9e^{-9y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise. Also $\mathbb{E}(Y^2) = \int_0^\infty (y^2)(9e^{-9y})\,dy$, and we use $u = y^2$ and $dv = 9e^{-9y}\,dy$, so $du = 2y\,dy$ and $v = -e^{-9y}$, to get $\mathbb{E}(Y^2) = -y^2e^{-9y}|_{y=0}^\infty - \int_0^\infty -2ye^{-9y}\,dy = \frac{2}{9}\int_0^\infty y\,9e^{-9y}\,dy$. We can either integrate a second time, by parts, or just recognize that the integral here is equal to $\mathbb{E}(Y)$, which is 1/9, as in the previous problem set, #4. So altogether $\mathbb{E}(Y^2) = \frac{2}{9}\mathbb{E}(Y) = (2/9)(1/9) = 2/81$. Thus $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/81 - (1/9)^2 = 1/81$.

Method #2: We compute $\mathbb{E}(Y^2) = \int_0^\infty \int_y^\infty (y^2)(18e^{-2x-7y})\,dx\,dy = \int_0^\infty (y^2)(-9e^{-2x-7y})|_{x=y}^\infty\,dy = \int_0^\infty (y^2)(9e^{-9y})\,dy$, and then everything else proceeds as in Method #1 above, i.e., we get $\mathbb{E}(Y^2) = 2/81$ in the same way from Method #1, starting on the second line. We also have $\mathbb{E}(Y) = 1/9$, so $\mathrm{Var}(Y) = 2/81 - (1/9)^2 = 1/81$.

5. We have $\mathbb{E}(X^2 + Y^3) = \mathbb{E}(X^2) + \mathbb{E}(Y^3)$.

As in the previous problem set, $f_X(x) = \frac{2}{9}(3-x)$ for $0 \le x \le 3$ and $f_X(x) = 0$ otherwise. So $\mathbb{E}(X^2) = \int_0^3 (x^2)(\frac{2}{9})(3-x)\,dx = \int_0^3 \frac{2}{9}(3x^2 - x^3)\,dx = \frac{2}{9}(x^3 - x^4/4)\big|_{x=0}^3 = 3/2$.

As in the previous problem set, $f_Y(y) = \frac{1}{2}(2-y)$ for $0 \le y \le 2$ and $f_Y(y) = 0$ otherwise. So $\mathbb{E}(Y^3) = \int_0^2 (y^3)(\frac{1}{2})(2-y)\,dy = \int_0^2 \frac{1}{2}(2y^3 - y^4)\,dy = \frac{1}{2}(2y^4/4 - y^5/5)\big|_{y=0}^2 = 4/5$.

Thus $\mathbb{E}(X^2 + Y^3) = \mathbb{E}(X^2) + \mathbb{E}(Y^3) = 3/2 + 4/5 = 23/10 = 2.3$.