STAT/MA 41600
Practice Problems: October 24, 2014
Solutions by Mark Daniel Ward

1. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{2}{9}(3-x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(X) = \int_0^3 \frac{2}{9}(3-x)x\,dx = \frac{2}{9}\int_0^3 (3x - x^2)\,dx = \frac{2}{9}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^{3} = \frac{2}{9}\left(\frac{27}{2} - \frac{27}{3}\right) = 1.$

2.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{1}{6}(4-x) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(X) = \int_0^2 \frac{1}{6}(4-x)x\,dx = \frac{1}{6}\int_0^2 (4x - x^2)\,dx = \frac{1}{6}\left(\frac{4x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^{2} = \frac{1}{6}\left(8 - \frac{8}{3}\right) = 8/9.$

b. As we saw earlier, the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{if } 0 \le y \le 6 \\ \frac{12-y}{54} & \text{if } 6 \le y \le 12 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(Y) = \int_0^6 \frac{1}{9}y\,dy + \int_6^{12} \frac{12-y}{54}y\,dy = \frac{1}{9}\frac{y^2}{2}\Big|_{y=0}^{6} + \int_6^{12} \frac{12y-y^2}{54}\,dy = \frac{1}{9}(18) + \frac{1}{54}\left(6y^2 - \frac{y^3}{3}\right)\Big|_{y=6}^{12} = 2 + \frac{1}{54}\left((864-576) - (216-72)\right) = 14/3.$

3.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using $u = x$ and $dv = 2e^{-2x}$ in integration by parts, we have $du = dx$ and $v = -e^{-2x}$, so we get $\mathbb{E}(X) = \int_0^\infty 2e^{-2x}x\,dx = -xe^{-2x}\big|_{x=0}^{\infty} - \int_0^\infty -e^{-2x}\,dx = \frac{e^{-2x}}{-2}\Big|_{x=0}^{\infty} = \frac{1}{2}.$

b. As we saw earlier, the density of Y is

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using $u = y$ and $dv = 7e^{-7y}$ in integration by parts, we have $du = dy$ and $v = -e^{-7y}$, so we get $\mathbb{E}(Y) = \int_0^\infty 7e^{-7y}y\,dy = -ye^{-7y}\big|_{y=0}^{\infty} - \int_0^\infty -e^{-7y}\,dy = \frac{e^{-7y}}{-7}\Big|_{y=0}^{\infty} = \frac{1}{7}.$

4.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{18}{7}\left(e^{-2x} - e^{-9x}\right) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\mathbb{E}(X) = \int_0^\infty \frac{18}{7}\left(e^{-2x} - e^{-9x}\right) x\,dx = \frac{18}{7}\left(\int_0^\infty e^{-2x}x\,dx - \int_0^\infty e^{-9x}x\,dx\right)$. Using $u = x$ and $dv = e^{-2x}$ in integration by parts, we have $du = dx$ and $v = -e^{-2x}/2$, so $\int_0^\infty e^{-2x}x\,dx = -xe^{-2x}/2\big|_{x=0}^\infty - \int_0^\infty -e^{-2x}/2\,dx = \left.\frac{e^{-2x}}{-4}\right|_{x=0}^\infty = 1/4$. Similarly, using $u = x$ and $dv = e^{-9x}$ in integration by parts, we have $du = dx$ and $v = -e^{-9x}/9$, so $\int_0^\infty e^{-9x}x\,dx = -xe^{-9x}/9\big|_{x=0}^\infty - \int_0^\infty -e^{-9x}/9\,dx = \left.\frac{e^{-9x}}{-81}\right|_{x=0}^\infty = 1/81$. Thus $\mathbb{E}(X) = \frac{18}{7}\left(\frac{1}{4} - \frac{1}{81}\right) = 11/18$.

b. As we saw earlier, the density of Y is

$$f_Y(y) = \begin{cases} 9e^{-9y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, using $u = y$ and $dv = 9e^{-9y}$ in integration by parts, we have $du = dy$ and $v = -e^{-9y}$, so we get $\mathbb{E}(Y) = \int_0^\infty 9e^{-9y}y\,dy = -ye^{-9y}\big|_{y=0}^\infty - \int_0^\infty -e^{-9y}\,dy = \left.\frac{e^{-9y}}{-9}\right|_{y=0}^\infty = \frac{1}{9}$.

5.

a. As we saw earlier, the density of X is

$$f_X(x) = \begin{cases} \frac{2}{9}(3 - x) & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

So, exactly as in Question 1, we have:

$\mathbb{E}(X) = \int_0^3 \frac{2}{9}(3 - x)x\,dx = \frac{2}{9}\int_0^3 (3x - x^2)\,dx = \frac{2}{9}\left.\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\right|_{x=0}^3 = \frac{2}{9}\left(\frac{27}{2} - 9\right) = 1.$

b. As we saw earlier, the density of X is

$$f_Y(y) = \begin{cases} \frac{1}{2}(2 - y) & \text{if } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{E}(Y) = \int_0^2 \frac{1}{2}(2 - y)y\,dy = \frac{1}{2}\int_0^2 (2y - y^2)\,dy = \frac{1}{2}\left.\left(y^2 - \frac{y^3}{3}\right)\right|_{y=0}^2 = \frac{1}{2}\left(4 - \frac{8}{3}\right) = 2/3.$