STAT/MA 41600
Practice Problems: October 22, 2014
Solutions by Mark Daniel Ward

1.

a. The density of Y, for $0 \le y \le 3$, is $f_Y(y) = \frac{2}{9}(3-y)$, as we saw in the previous problem set. The joint density is 2/9. Thus, the conditional density of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2/9}{\frac{2}{9}(3-y)} = \frac{1}{3-y}, \qquad \text{for } 0 \le x \le 3-y,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. The conditional probability is $P(X \le 1 \mid Y = 1) = \int_0^1 f_{X|Y}(x \mid 1)\,dx = \int_0^1 \frac{1}{3-1}\,dx = 1/2$.

c. Using Bayes' Theorem, we have $P(X \le 1 \mid Y \le 1) = \frac{P(X \le 1 \text{ and } Y \le 1)}{P(Y \le 1)} = \frac{1/(9/2)}{(5/2)/(9/2)} = 2/5$.

2.

a. The density of Y, for $0 \le y \le 6$, is $f_Y(y) = 1/9$, as we saw in the previous problem set. The joint density is 1/18. Thus, the conditional density of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/18}{1/9} = 1/2, \qquad \text{for } 0 \le x \le 2,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. The density of Y, for $6 \le y \le 12$, is $f_Y(y) = (12-y)/54$, as we saw in the previous problem set. The joint density is 1/18. Thus, the conditional density of X given Y is

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/18}{(12-y)/54} = 3/(12-y), \qquad \text{for } 0 \le x \le (12-y)/3,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

c. Using Bayes' Theorem, $P(X \le 1 \mid 3 \le Y \le 9) = \frac{P(X \le 1 \text{ and } 3 \le Y \le 9)}{P(3 \le Y \le 9)} = \frac{6/18}{10.5/18} = 4/7$.

3.

a. Since X and Y are independent, then $f_{X|Y}(x \mid y) = f_X(x)$. Thus $f_{X|Y}(x \mid y) = 2e^{-2x}$ for $x > 0$, and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. Since X and Y are independent, then $P(X \ge 1 \mid Y = 3) = P(X \ge 1) = \int_1^\infty 2e^{-2x}\,dx = -e^{-2x}\big|_{x=1}^{\infty} = e^{-2} = 0.1353$.

c. Since X and Y are independent, then then $f_{Y|X}(y \mid x) = f_Y(y)$. Thus $f_{Y|X}(y \mid x) = 7e^{-7y}$ for $y > 0$, and $f_{Y|X}(y \mid x) = 0$ otherwise. So $P(Y \le 1/5 \mid X = 2.7) = P(Y \le 1/5) = \int_0^{1/5} 7e^{-7y}\,dx = -e^{-7y}\big|_{y=0}^{1/5} = 1 - e^{-7/5} = 0.7534$.

4.

a. The density of $f_Y(y)$, for $y > 0$, is $f_Y(y) = \int_y^\infty 18e^{-2x-7y}\,dx = -9e^{-2x-7y}\big|_{x=y}^{\infty} = 9e^{-9y}$. So

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{18e^{-2x-7y}}{9e^{-9y}} = 2e^{-2x+2y} \qquad \text{if } x > y,$$

and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. The density of $f_X(x)$, for $x > 0$, is $f_X(x) = \int_0^x 18e^{-2x-7y}\,dy = \frac{18}{7}\left(e^{-2x} - e^{-9x}\right)$. So

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{18e^{-2x-7y}}{\frac{18}{7}\left(e^{-2x} - e^{-9x}\right)} = \frac{7e^{-7y}}{(1 - e^{-7x})} \qquad \text{if } 0 < y < x,$$

and $f_{Y|X}(y \mid x) = 0$ otherwise.

5.

a. Since X and Y are independent, then $f_{X|Y}(x \mid y) = f_X(x)$. Thus $f_{X|Y}(x \mid y) = \frac{2}{9}(3-x)$ for $0 \le x \le 3$, and $f_{X|Y}(x \mid y) = 0$ otherwise.

b. Since X and Y are independent, $P(X \le 2 \mid Y = 3/2) = P(X \le 2) = \int_0^2 \frac{2}{9}(3-x)\,dx = \frac{2}{9}\left(3x - \frac{x^2}{2}\right)\Big|_{x=0}^{2} = 8/9$.

c. Since X and Y are independent, $P(Y \ge 1 \mid X = 5/4) = P(Y \ge 1) = \int_1^2 \frac{1}{2}(2-y)\,dy = \frac{1}{2}\left(2y - \frac{y^2}{2}\right)\Big|_{y=1}^{2} = 1/4$.