STAT/MA 41600
Practice Problems: October 6, 2014
Solutions by Mark Daniel Ward

1. Hearts.

a. *Method #1:* We win if 2 or 3 of the 3 cards are hearts. So the probability of winning is $\frac{\binom{39}{1}\binom{13}{2}}{\binom{52}{3}} + \frac{\binom{13}{3}}{\binom{52}{3}} = 64/425 = 0.1506$.
Method #2: The probability of winning is 1 minus the probability of losing. We lose if 0 or 1 of the 3 cards are hearts. So the probability of winning is $1 - \frac{\binom{39}{2}\binom{13}{1}}{\binom{52}{3}} - \frac{\binom{39}{3}}{\binom{52}{3}} = 64/425 = 0.1506$.

b. Put the cards in a row (e.g., before you look at them). Let X_1, X_2, X_3 indicate whether the first, second, third card are hearts. The total number of hearts is $X_1 + X_2 + X_3$. Also $\mathbb{E}(X_j) = 1/4$ for each j. So the expected number of hearts is $\mathbb{E}(X_1 + X_2 + X_3) = 1/4 + 1/4 + 1/4 = 3/4$.

2. Socks.

a. There are $\binom{21+8+4}{6} = 1107568$ equally-likely ways to pick the socks. There are $\binom{21}{2}\binom{8}{2}\binom{4}{2} = 35280$ ways to get 2 of each color. So the probability is $35280/1107568 = 0.0319$.

b. All of the socks are white with probability $\binom{21}{6}/\binom{33}{6} = 969/19778$. All of the socks are black with probability $\binom{8}{6}/\binom{33}{6} = 1/39556$. These possibilities are disjoint. We cannot get 6 black socks. So the total probability of getting 6 socks of the same color is $969/19778 + 1/39556 = 1939/39556 = 0.0490$.

c.

socks	probability
2 white, 4 black	$\binom{21}{2}\binom{8}{4}/\binom{33}{6} = 525/39556$
2 white, 4 brown	$\binom{21}{2}\binom{4}{4}/\binom{33}{6} = 15/79112$
2 black, 4 white	$\binom{8}{2}\binom{21}{4}/\binom{33}{6} = 5985/39556$
2 black, 4 brown	$\binom{8}{2}\binom{4}{4}/\binom{33}{6} = 1/39556$
2 brown, 4 white	$\binom{4}{2}\binom{21}{4}/\binom{33}{6} = 2565/79112$
2 brown, 4 black	$\binom{4}{2}\binom{8}{4}/\binom{33}{6} = 15/39556$

The total probability (add the probabilities of these disjoint events) is $1954/9889 = 0.1976$.

3. Married couples.

Let X_j indicate if the jth man sits across from his wife. Then $\mathbb{E}(X_j) = 1/19$, since no matter where the man sits, his wife can sit in 19 other possible chairs, only 1 of which is directly across from him. Thus the expected number of men sitting across from their wives is $\mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10}) = 1/19 + \cdots + 1/19 = 10/19$.

4. Ramen Noodles.

a. We either need 1 beef and 2 chickens, which happens with probability $\frac{\binom{10}{1}\binom{10}{2}}{\binom{20}{3}} = 15/38$, or 2 beef and 1 chicken, which also happens with probability $15/38$. So the probability of at least one of each is $15/38 + 15/38 = 15/19$.

b. This is the complementary probability, i.e., $1 - 15/19 = 4/19$. If you prefer, you can calculate this directly, because the probability of 3 beefs is $\binom{10}{3}/\binom{20}{3} = 2/19$, and the probability of 3 chickens is $2/19$, so again, the probability of three of the same flavor is $2/19 + 2/19 = 4/19$.

5. Picking letters at random.

a. There are $26^5 = 11881376$ ways that they could pick the letters, and there are $26!/21! = (26)(25)(24)(23)(22) = 7893600$ ways that the choices are unique. So the probability that they are unique is $7893600/11881376 = 18975/28561 = 0.6644$.

b. As before, there are $26^5 = 11881376$ equally-likely ways that they can make their choices.

Method #1: The number of ways that these are unique is $26!/21! = (26)(25)(24)(23)(22) = 7893600$, and for each of these possibilities, only 1 out of every $5!$ are in increasing order. So the probability that they are in increasing order is $\frac{7893600/5!}{11881376} = 1265/228488 = 0.005536$.

Method #2: There are $\binom{26}{5}$ ways to choose the letters without repeats, and once they are chosen, there is only 1 way to put them in increasing order. So the probability is $\binom{26}{5}/11881376 = 1265/228488 = 0.005536$.