STAT/MA 41600
Practice Problems: September 17, 2014
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1. Butterflies. The mass of X is

$$p_X(0) = 0.3424, \qquad p_X(1) = 0.4644, \qquad p_X(2) = 0.1741, \qquad p_X(3) = 0.0191,$$

so the expected value of X is

$$\mathbb{E}(X) = (0)(0.3424) + (1)(0.4644) + (2)(0.1741) + (3)(0.0191) = .87$$

2. Dependence/independence among dice rolls. We have $P(X = j) = (5/6)^{j-1}(1/6)$ for each positive integer j. So

$$\mathbb{E}(X) = (1)P(X = 1) + (2)P(X = 2) + (3)P(X = 3) + \cdots$$
$$= \sum_{j=1}^{\infty} j(5/6)^{j-1}(1/6)$$
$$= \frac{1}{6} \sum_{j=1}^{\infty} j(5/6)^{j-1}$$

We notice that $j(5/6)^{j-1}$ is the derivative of x^j (with respect to x), evaluated at $x = 5/6$. So we rewrite the equation above as follows:

$$\mathbb{E}(X) = \frac{1}{6} \sum_{j=1}^{\infty} \frac{d}{dx} x^j \Big|_{x=5/6} = \frac{1}{6} \frac{d}{dx} \sum_{j=1}^{\infty} x^j \Big|_{x=5/6}$$

We use our scrap paper to compute:

$$\sum_{j=1}^{\infty} x^j = x \sum_{j=0}^{\infty} x^j = \frac{x}{1-x}$$

So we get

$$\mathbb{E}(X) = \frac{1}{6} \frac{d}{dx} \frac{x}{1-x} \Big|_{x=5/6}$$
$$= \frac{1}{6} \frac{(1-x)(1) - (x)(-1)}{(1-x)^2} \Big|_{x=5/6}$$
$$= \frac{1}{6} \frac{1}{(1-x)^2} \Big|_{x=5/6}$$
$$= \frac{1}{6} \frac{1}{\left(1 - \frac{5}{6}\right)^2}$$
$$= \frac{1}{6} \frac{1}{(1/6)^2}$$
$$= 6$$

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So the student expects to roll 6 times to get the first 4.

The same reasoning shows that $\mathbb{E}(Y) = 6$ too.

3. Wastebasket basketball. As in Problem Set 8, The mass of X is

$$p_X(1) = 1/3, \quad p_X(2) = 2/9, \quad p_X(3) = 4/27, \quad p_X(4) = 8/81, \quad p_X(5) = 16/243, \quad p_X(6) = 32/243.$$

So the expected value of X is

$$\mathbb{E}(X) = (1)(1/3)+(2)(2/9)+(3)(4/27)+(4)(8/81)+(5)(16/243)+(6)(32/243) = 665/243 = 2.7366$$

4. Two 4-sided dice. We compute

$$\begin{aligned}
\mathbb{E}(X) &= (1)p_{X,Y}(1,1) + (1)p_{X,Y}(1,2) + (1)p_{X,Y}(1,3) + (1)p_{X,Y}(1,4) \\
&\quad + (2)p_{X,Y}(2,2) + (2)p_{X,Y}(2,3) + (2)p_{X,Y}(2,4) \\
&\quad\quad + (3)p_{X,Y}(3,3) + (3)p_{X,Y}(3,4) \\
&\quad\quad\quad + (4)p_{X,Y}(4,4) \\
&= (1)(1/16) + (1)(2/16) + (1)(2/16) + (1)(2/16) \\
&\quad + (2)(1/16) + (2)(2/16) + (2)(2/16) \\
&\quad\quad + (3)(1/16) + (3)(2/16) \\
&\quad\quad\quad + (4)(1/16) \\
&= 30/16 \\
&= 15/8
\end{aligned}$$

We compute

$$\begin{aligned}
\mathbb{E}(Y) &= (1)p_{X,Y}(1,1) + (2)p_{X,Y}(1,2) + (3)p_{X,Y}(1,3) + (4)p_{X,Y}(1,4) \\
&\quad + (2)p_{X,Y}(2,2) + (3)p_{X,Y}(2,3) + (4)p_{X,Y}(2,4) \\
&\quad\quad + (3)p_{X,Y}(3,3) + (4)p_{X,Y}(3,4) \\
&\quad\quad\quad + (4)p_{X,Y}(4,4) \\
&= (1)(1/16) + (2)(2/16) + (3)(2/16) + (4)(2/16) \\
&\quad + (2)(1/16) + (3)(2/16) + (4)(2/16) \\
&\quad\quad + (3)(1/16) + (4)(2/16) \\
&\quad\quad\quad + (4)(1/16) \\
&= 50/16 \\
&= 25/8
\end{aligned}$$

5. Pick two cards. From Problem Set 7, we know that the mass of X is

$$P(X = 0) = 30/51, \qquad P(X = 1) = 80/221, \qquad P(X = 2) = 11/221.$$

So

$$\mathbb{E}(X) = (0)(30/51) + (1)(80/221) + (2)(11/221) = 6/13.$$

Let A_1, A_2 be (respectively) the events that the first, second card is a 10 card. Even though the cards appear simultaneously, we can just randomly treat one of them as the first and the other as the second. So

$$P(Y = 0) = P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c \mid A_1^c) = (48/52)(47/51) = 188/221.$$

Using the same A_1, A_2 as above,

$$P(Y = 1) = P(A_1 \cap A_2^c) + P(A_1^c \cap A_2) = (4/52)(48/51) + (48/52)(4/51) = 32/221.$$

Using the same A_1, A_2 as above,

$$P(Y = 2) = P(A_1 \cap A_2) = (4/52)(3/51) = 1/221.$$

So

$$\mathbb{E}(Y) = (0)(188/221) + (1)(32/221) + (2)(1/221) = 2/13.$$