Moment Generating Function of a Poisson random variable with $E(X)=\lambda$.

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} e^{e^t\lambda}$$

$(e^t\lambda)^x$

$$= e^{(e^t-1)\lambda}$$

If we take a derivative with respect to t and evaluate at $t=0$, we get

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = M_X'(0) = \lambda \quad \checkmark$$

If we take two derivatives with respect to t and evaluate at $t=0$, get

$$E(X^2) = \frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0} = M_X''(0) = \lambda^2 + \lambda$$

So $\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda \quad \checkmark$