Example Order Statistics for uniform (continuous) random variables

Let U_1, U_2, U_3, U_4 be indep, continuous, uniforms on $[0,5]$.

Then $U_{(3)}$ denotes the 3rd order statistic of these random variables,

i.e. $U_{(3)}$ = 3rd smallest = 2nd largest of U_1, U_2, U_3, U_4.

The CDF of $U_{(3)}$: For $0<a<5$

$$
\begin{aligned}
F_{U_{(3)}}(a) = P(U_{(3)} \le a) = \ & P(U_1, \dots, U_4 \le a) \\
&+ P(U_1, U_2, U_3 \le a,\ a < U_4 < 5) \\
&+ P(U_1, U_2, U_4 \le a,\ a < U_3 < 5) \\
&+ P(U_1, U_3, U_4 \le a,\ a < U_2 < 5) \\
&+ P(U_2, U_3, U_4 \le a,\ a < U_1 < 5) \\
= \ & \left(\tfrac{a}{5}\right)^4 + 4\left(\tfrac{a}{5}\right)^3\left(1 - \tfrac{a}{5}\right) \quad \text{for } 0<a<5
\end{aligned}
$$

So the density of $U_{(3)}$ is

$$f_{U_{(3)}}(u) = \frac{\partial}{\partial u}\left[\left(\tfrac{u}{5}\right)^4 + 4\left(\tfrac{u}{5}\right)^3\left(1-\tfrac{u}{5}\right)\right] = (12)\left(\tfrac{1}{5}\right)\left(\tfrac{u}{5}\right)^2\left(1-\tfrac{u}{5}\right).$$

for $0<u<5$.

We could (alternatively) have just used the general formula for the density of an order statistic:

$$f_{U_{(3)}}(u) = \underbrace{\binom{4}{2,1,1}}_{=\frac{4!}{2!1!1!} = 12} \underbrace{\left(\tfrac{1}{5}\right)}_{\text{density of unif }(0,5)} \underbrace{\left(\tfrac{u}{5}\right)}_{\text{CDF of a unif }(0,5)}^2 \underbrace{\left(1-\tfrac{u}{5}\right)}_{\text{Complement of the CDF of a unif }(0,5)}^1 \quad \text{for } 0<u<5.$$

We could check that this is a valid density function:

$$\int_0^5 12\left(\tfrac{1}{5}\right)\left(\tfrac{u}{5}\right)^2\left(1-\tfrac{u}{5}\right) du = 1$$

also $E(U_{(3)}) = \int_0^5 (u)(12)\left(\tfrac{1}{5}\right)\left(\tfrac{u}{5}\right)^2\left(1-\tfrac{u}{5}\right) du = 3.$

In fact, 0 (left hand interval), $E(U_{(1)})$, $E(U_{(2)})$, $E(U_{(3)})$, $E(U_{(4)})$, 5 = right hand interval are evenly spaced among $[0,5]$