<u>Example</u> Suppose X_1 and X_2 are independent exponential random variables with $E(X_1) = E(X_2) = \frac{1}{3}$.

Write $X_{(1)}$ to denote $\min(X_1, X_2)$ ← 1st order statistic

$X_{(2)}$ to denote $\max(X_1, X_2)$ ← 2nd order statistic

For $0 < a < b$ find $F_{X_{(1)}, X_{(2)}}(a, b)$.

$= P(X_{(1)} \leq a,\ X_{(2)} \leq b)$

$$= \underbrace{(1-e^{-3a})(1-e^{-3a})}_{\text{prob both } X_1, X_2 \text{ smaller than } a} + \underbrace{(1-e^{-3a})}_{\text{prob } X_1 < a}\underbrace{(e^{-3a}-e^{-3b})}_{\text{prob } a < X_2 < b} + \underbrace{(e^{-3a}-e^{-3b})}_{\text{prob } a < X_1 < b}\underbrace{(1-e^{-3a})}_{\text{prob } X_2 < a}$$

$$= (1-e^{-3a})^2 + 2(1-e^{-3a})(e^{-3a}-e^{-3b})$$

Find the joint density of $X_{(1)}, X_{(2)}$

$$f_{X_{(1)}, X_{(2)}}(x_1, x_2) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\left[(1-e^{-3x_1})^2 + 2(1-e^{-3x_1})(e^{-3x_1}-e^{-3x_2})\right]$$

$$= (2)(3e^{-3x_1})(3e^{-3x_2}) \quad \text{for } 0 < x_1 < x_2.$$

This is a special case of a more general idea

Find the density of $X_{(1)} = \min(X_1, X_2)$ = 1st order statistic

$$f_{X_{(1)}}(x_1) = \int_{x_1}^{\infty} (2)(3e^{-3x_1})(3e^{-3x_2})\,dx_2 = 6e^{-6x_1} \quad \text{for } x_1 > 0$$

So $X_{(1)}$ is an exponential random variable with $E(X_{(1)}) = \frac{1}{6}$.

Find the density of $X_{(2)} = \max(X_1, X_2)$ = 2nd order statistic

$$f_{X_{(2)}}(x_2) = \int_0^{x_2} (2)(3e^{-3x_1})(3e^{-3x_2})\,dx_1 = 6e^{-3x_2} - 6e^{-6x_2}$$

$$= 6e^{-3x_2}(1-e^{-3x_2}) \quad \text{for } x_2 > 0$$

So $X_{(2)}$ is not an exponential random variable.

For those who are curious, $X_{(2)} - X_{(1)}$ is an exponential random variable, but we won't show it here.