Example with continuous random variables

Say X, Y have joint probability density $f_{X,Y}(x,y) = 40e^{-3x-5y}$ for $0<x<y$
$= 0$ otherwise.

Find $Var(X+Y)$.

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

$$E(X^2) - (E(X))^2$$

$$\int_0^\infty \int_x^\infty (x^2)(40e^{-3x-5y})\,dy\,dx = \frac{1}{32} \qquad \left(\int_0^\infty \int_x^\infty (x)(40e^{-3x-5y})\,dy\,dx\right)^2 = \left(\frac{1}{8}\right)^2$$

$$Var(X) = \frac{1}{32} - \left(\frac{1}{8}\right)^2.$$

$$Var(Y) = E(Y^2) - (E(Y))^2$$

$$\int_0^\infty \int_x^\infty (y^2)(40e^{-3x-5y})\,dy\,dx = \frac{129}{800} \qquad \left(\int_0^\infty \int_x^\infty (y)(40e^{-3x-5y})\,dy\,dx\right)^2 = \left(\frac{13}{40}\right)^2$$

$$Var(Y) = \frac{129}{800} - \left(\frac{13}{40}\right)^2$$

$$Cov(X,Y) = E(XY) - \underbrace{E(X)}_{\frac{1}{8}}\,\underbrace{E(Y)}_{\frac{13}{40}}$$

$$\int_0^\infty \int_x^\infty (xy)(40e^{-3x-5y})\,dy\,dx = \frac{9}{160} \qquad Cov(X,Y) = \frac{9}{160} - \left(\frac{1}{8}\right)\left(\frac{13}{40}\right)$$

Put it all together:

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y) = \frac{41}{400}.$$