Hat problem: Suppose 10 people attend a party, each checks their hat at the door, they pick up a random hat when they leave, all choices equally likely.

Let X indicate whether Alice get her back, i.e. $X=1$ if she does, $X=0$ otherwise.

Let Y indicate whether Bob gets his hat back: $Y=1$ if so, $Y=0$ otherwise.

Find $Var(X+Y)$

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$
$$= \left(\tfrac{1}{10}\right)\left(\tfrac{9}{10}\right) + \left(\tfrac{1}{10}\right)\left(\tfrac{9}{10}\right) + 2\left(\left(\tfrac{1}{10}\right)\left(\tfrac{1}{9}\right) - \left(\tfrac{1}{10}\right)\left(\tfrac{1}{10}\right)\right)$$
$$= \frac{41}{225}$$

$$Cov(X,Y) = \left(\tfrac{1}{10}\right)\left(\tfrac{1}{9}\right) - \left(\tfrac{1}{10}\right)\left(\tfrac{1}{10}\right)$$
$$E(XY) - \underbrace{E(X)}_{\frac{1}{10}}\underbrace{E(Y)}_{\frac{1}{10}}$$
$$E(XY) = P(XY=1)(1) + \cancel{P(XY=0)(0)}$$
$$= P(X=1 \text{ and } Y=1)$$
$$= P(X=1)P(Y=1 \mid X=1) = \left(\tfrac{1}{10}\right)\left(\tfrac{1}{9}\right)$$

Extend: Let $X_j = 1$ if jth person gets her/his hat back, $X_j = 0$ otherwise

So $X_1 + \dots + X_{10}$ is the total # of people who get their own hat back.

$$E(X_1 + \dots + X_{10}) = E(X_1) + \dots + E(X_{10}) = \tfrac{1}{10} + \dots + \tfrac{1}{10} = 1.$$

$$Var(X_1 + \dots + X_{10}) = \sum_{i=1}^{10} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$
$$= (10)\left(\tfrac{1}{10}\right)\left(\tfrac{9}{10}\right) + (90)\left(\left(\tfrac{1}{10}\right)\left(\tfrac{1}{9}\right) - \left(\tfrac{1}{10}\right)\left(\tfrac{1}{10}\right)\right)$$
$$= 1$$

all 100 pairs except those 10 on the diagonal handled here