Suppose X is any Normal random variable with parameters μ_X and σ_X^2. Claim that X and $\sigma_X Z + \mu_X$ have the same distribution. Why?

$$P(\sigma_X Z + \mu_X \le a) = P(\sigma_X Z \le a - \mu_X)$$

$$\underset{F_{\sigma_X Z + \mu_X}(a)}{''} \quad = P\left(Z \le \frac{a-\mu_X}{\sigma_X}\right) = \int_{-\infty}^{\frac{a-\mu_X}{\sigma_X}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

sub $u = \sigma_X z + \mu_X$

$$\frac{u-\mu_X}{\sigma_X} = z$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{u-\mu_X}{\sigma_X}\right)^2/2} \frac{du}{\sigma_X}$$

$$= \int_{-\infty}^{a} \boxed{\frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(u-\mu_X)^2/(2\sigma_X)^2}}\, du$$

↑ same as density of X

2σ²

So indeed $\sigma_X Z + \mu_X$ and X have the same distribution.

So $\sigma_X Z + \mu_X$ is normal too.

$$E(X) = E(\sigma_X Z + \mu_X) = \sigma_X E(Z) + \mu_X = \mu_X$$

$$Var(X) = Var(\sigma_X Z + \mu_X) = \sigma_X^2\, Var(Z) = \sigma_X^2.$$