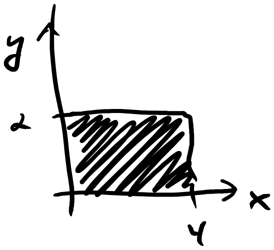
Another example: Suppose X, Y have joint density $f_{X,Y}(x,y) = \frac{x}{16}(2-y)$ for $0<x<4$, $0<y<2$
$= 0$ otherwise.



Find conditional density of Y, given $X=3$:

$$f_{Y|X}(y|3) = \frac{f_{X,Y}(3,y)}{f_X(3)} = \frac{\frac{3}{16}\cdot(2-y)}{3/8} = \frac{1}{2}(2-y)$$

Need: $f_X(3) = \int_0^2 f_{X,Y}(3,y)\,dy = \int_0^2 \frac{3}{16}(2-y)dy = \frac{3}{16}\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^{2}$

$= \frac{3}{16}(4-2) = \frac{6}{16} = \frac{3}{8}$

Check that $f_{Y|X}(y|3)$ is a valid conditional density:

$$\int_0^2 \frac{1}{2}(2-y)\,dy = \frac{1}{2}\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^{2} = \frac{1}{2}(4-2) = 1 \checkmark$$

Find $P(Y \le 3/4 \mid X=3) = \int_0^{3/4} \frac{1}{2}(2-y)dy = \frac{1}{2}\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^{3/4}$

$= \frac{1}{2}\left(\frac{3}{2} - \frac{9/16}{2}\right)$

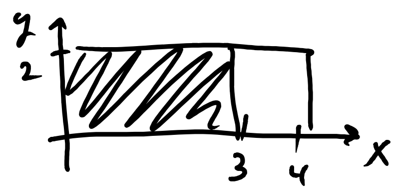
$= \frac{1}{2}\left(\frac{48}{32} - \frac{9}{32}\right) = \frac{1}{2}\left(\frac{39}{32}\right) = \frac{39}{64}.$

change $X=3$ to $X \le 3$

Q: Suppose $P(Y \le 3/4 \mid X \le 3) = \frac{P(Y \le 3/4 \text{ and } X \le 3)}{P(X \le 3)}$

No conditional density needed!

$f_{X,Y}(x,y) = \frac{x}{16}(2-y) \quad 0<x<4, \; 0<y<2$



$P(X \le 3) = \int_0^2 \int_0^3 \frac{x}{16}(2-y)\,dx\,dy$

$= \int_0^2 \frac{x^2}{32}(2-y)\Big|_{x=0}^{3} dy$

$= \int_0^2 \frac{9}{32}(2-y)dy = \frac{9}{32}\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^{2}$

$= \frac{9}{32}(4-2) = \frac{18}{32} = \frac{9}{16}.$

$P(Y \le \frac{3}{4} \text{ and } X \le 3) = \int_0^{3/4}\int_0^3 \frac{x}{16}(2-y)dx\,dy = \int_0^{3/4} \frac{9}{32}(2-y)\,dy$

$= \frac{9}{32}\left(2y - \frac{y^2}{2}\right)\Big|_{y=0}^{3/4}$

$= \frac{9}{32}\left(\frac{3}{2} - \frac{9/16}{2}\right) = \frac{9}{32}\left(\frac{48}{32} - \frac{9}{32}\right) = \frac{9}{32}\cdot\frac{39}{32}$

$P(Y \le 3/4 \mid X \le 3) = \frac{\cancel{\frac{9}{32}}\cdot\frac{39}{32}(1/2)}{\cancel{9/16}\cdot\cancel{(1/2)}} = 39/64.$

$P(Y \le 3/4 \mid X=3) = 39/64$ too! Not surprising because X, Y indep!

$f_{X,Y}(x,y) = \frac{x}{16}\cdot(2-y)$ factored (x part)·(y part) on a rectangle.