Independent continuous random variables

Several ways to specify whether two random variables X and Y are independent. All of these ways are equivalent to each other.

One way is to show that the joint density (i.e., joint probability density function) of X and Y can be factored into two parts, one of which only has x's and the other of which only has y's. With some possible rescaling, this means that the two parts are the densities of X and Y respectively, and it also means that X and Y are independent continuous random variables. In other words, X and Y are independent continuous random variables if and only if their joint density can be factored into a product of their (single variables) densities:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y.$$

Another possible formulation of independence (which is equivalent to the one above) is that the joint CDF (joint cumulative distribution function) factors into a product of two functions, one of which only has x's, and the other of which only has y's. Again, with some possible rescaling, the result is that the joint CDF factors into the CDF of X times the CDF of Y. I.e., X and Y are independent if and only if

$$F_{X,Y}(x, y) = F_X(x) F_Y(y) \qquad \text{for all } x, y.$$

One more equivalent interpretation. We have not (yet) said what the conditional density of a continuous random variable is, but once we learn it, we can formulate independence in terms of the conditional density. Namely, X and Y are independent continuous random variables if and only if

$$f_{X|Y}(x|y) = f_X(x) \qquad \text{for all } x, y,$$

or equivalently,

$$f_{Y|X}(y|x) = f_Y(y) \qquad \text{for all } x, y.$$