Using derivatives to find expected value and variance of Geometric random variables. Say X is a Geometric(p) random variable.

$$E(X) = \sum_{j=1}^{\infty} j\,P(X=j) = \sum_{j=1}^{\infty} j q^{j-1} p = p \sum_{j=1}^{\infty} \frac{d}{dq} q^j = p \frac{d}{dq}\left(\sum_{j=1}^{\infty} q^j\right) = p \frac{d}{dq} \frac{q}{1-q}$$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}$$

note $\quad j q^{j-1} = \frac{d}{dq} q^j$

quotient rule: $\quad \frac{d}{dq} \frac{q}{1-q} = \frac{(1-q)(1) - (q)(-1)}{(1-q)^2} = \frac{1}{(1-q)^2} = \frac{1}{p^2}$

$$Var(X) = E(\underbrace{X^2}_{E((X)(X-1)+X)}) - (\underbrace{E(X)}_{1/p})^2 = E((X)(X-1)) + E(X) - (E(X))^2$$

$$+\ 1/p - (1/p)^2$$

$$E((X)(X-1)) = \sum_{j=1}^{\infty} (j)(j-1)\underbrace{P(X=j)}_{q^{j-1}\cdot p} = pq \sum_{j=1}^{\infty} \underbrace{\boxed{(j)(j-1)q^{j-2}}}_{=\frac{d^2}{dq^2} q^j} = pq \sum_{j=1}^{\infty} \frac{d^2}{dq^2} q^j$$

$$= pq \frac{d^2}{dq^2} \sum_{j=1}^{\infty} q^j$$

$$= pq \frac{d^2}{dq^2} \frac{q}{1-q}$$

$$= pq \frac{d}{dq} \frac{1}{(1-q)^2}$$

$$= pq(-2)(1-q)^{-3}\cdot(-1)$$

$$= \frac{2pq}{(1-q)^3}$$

$$= \frac{2pq}{p^3}$$

$$= \frac{2q}{p^2}.$$

$$Var(X) = E((X)(X-1)) + E(X) - (E(X))^2$$

$$= \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2$$

$$= \frac{2(1-p) + p - 1}{p^2}$$

$$= \frac{1-p}{p^2}$$

$$= \frac{q}{p^2}.$$