Consider two random variables X and Y that are indicator random variables for (respectively) events A and B. Recall that this means $X=1 \leftrightarrow A$ occurs, $X=0$ otherwise. $Y=1 \leftrightarrow B$ occurs, $Y=0$ otherwise.

Idea: A and B are independent events if and only if X, Y independent random variables.

$p_{X,Y}(1,1) = P(A \cap B)$ equal if and only if A, B indep.

$p_X(1) p_Y(1) = P(A)P(B)$

$p_{X,Y}(1,0) = P(A \cap B^c)$ equal if and only if A, B independent

$p_X(1) p_Y(0) = P(A)P(B^c)$

$p_{X,Y}(0,0) = P(A^c \cap B^c)$ equal iff A, B indep

$p_X(0) p_Y(0) = P(A^c)P(B^c)$

$p_{X,Y}(0,1) = P(A^c \cap B)$ equal iff A, B indep.

$p_X(0) p_Y(1) = P(A^c)P(B)$

So in summary, random variables X, Y are independent exactly when events A, B are independent.