STAT/MA 41600
In-Class Problem Set #32: October 26, 2018
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Problem Set 32 Answers

1. We compute $P(X > U) = \int_0^{10}\int_u^{\infty}(1/10)(1/5)e^{-x/5}\,dx\,du = \int_0^{10}(1/10)(-e^{-x/5})|_{x=u}^{\infty}\,du = \int_0^{10}(1/10)(e^{-u/5})\,du = (-1/2)(e^{-u/5})|_{u=0}^{10} = (1/2)(1-e^{-2}) = 0.4323.$

2a. We compute $P(|X-8| \le 2) = P(6 \le X \le 10) = \int_6^{10}(1/10)e^{-x/10}\,dx = -e^{-x/10}|_{x=6}^{10} = e^{-3/5} - e^{-1} = 0.1809.$

2b. We compute $P(X \ge 12 \mid X \ge 9) = \frac{P(X\ge 12\ \&\ X\ge 9)}{P(X\ge 9)} = \frac{P(X\ge 12)}{P(X\ge 9)} = \frac{e^{-12/10}}{e^{-9/10}} = e^{-3/10} = 0.7408.$

Alternatively, we can use the memoryless property of exponential random variables, so we get $P(X \ge 12 \mid X \ge 9) = P(X \ge 3) = e^{-3/10} = 0.7408.$

3. We let X, Y, Z denote the times (in seconds) until the phone rings, email arrives, or computer beeps. Then $P(Y \le X\ \&\ Y \le Z) = \int_0^{\infty}\int_y^{\infty}\int_y^{\infty}(1/30)e^{-x/30}(1/20)e^{-y/20}(1/15)e^{-z/15}\,dz\,dx\,dy = \frac{1}{20}\int_0^{\infty}(-e^{-x/30})|_{x=y}^{\infty}e^{-y/20}(-e^{-z/15})|_{z=y}^{\infty}\,dy = \frac{1}{20}\int_0^{\infty}e^{-y/30}e^{-y/20}e^{-y/15}\,dy = \frac{1}{20}\int_0^{\infty}(e^{-3y/20})\,dy = (-1/3)(e^{-3y/20})|_{y=0}^{\infty} = 1/3.$

Alternatively, we can let W denote the minimum of X and Z. Since the minimum of independent exponential random variables is also an exponential random variable, and we add the parameters, then W is an exponential random variable with parameter $\lambda = 1/30 + 1/15 = 1/10$. So we get $P(Y \le X\ \&\ Y \le Z) = P(Y \le W) = \int_0^{\infty}\int_y^{\infty}(1/20)e^{-y/20}(1/10)e^{-w/10}\,dw\,dy = \frac{1}{20}\int_0^{\infty}e^{-y/20}(-e^{-w/10})|_{w=y}^{\infty}dy = \frac{1}{20}\int_0^{\infty}e^{-y/20}(e^{-y/10})dy = \frac{1}{20}\int_0^{\infty}e^{-3y/20}dy = (-1/3)e^{-3y/20}|_{y=0}^{\infty} = 1/3.$

4. We compute

$$\begin{aligned}
P(Y/2 < X < Y) &= \int_0^{\infty}\int_{y/2}^{y}(1/3)e^{-x/3}(1/4)e^{-y/4}\,dx\,dy \\
&= \int_0^{\infty}(-e^{-x/3})|_{x=y/2}^{y}(1/4)e^{-y/4}\,dy \\
&= \int_0^{\infty}(e^{-y/6} - e^{-y/3})(1/4)e^{-y/4}\,dy \\
&= \int_0^{\infty}(e^{-5y/12} - e^{-7y/12})(1/4)\,dy \\
&= ((-12/5)e^{-5y/12} - (-12/7)e^{-7y/12})(1/4)|_{y=0}^{\infty} \\
&= (12/5 - 12/7)(1/4) \\
&= 6/35 \\
&= 0.1714.
\end{aligned}$$