STAT/MA 41600
In-Class Problem Set #22: October 1, 2018
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Problem Set 22 Answers

1a. Let $X_j = 1$ if the jth person gets the correct plate of food, and $X_j = 0$ otherwise. Then the expected number of students who get their own proper plates of food is $\mathbb{E}(X_1 + \cdots + X_8) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_8) = 8\mathbb{E}(X_1) = (8)(1/8) = 1$.
1b. We compute $\mathbb{E}((X_1 + \cdots + X_8)^2) = 8\mathbb{E}(X_1X_1) + 56\mathbb{E}(X_1X_2)$. As before, we have $\mathbb{E}(X_1X_1) = \mathbb{E}(X_1) = 1/8$. We also notice that $\mathbb{E}(X_1X_2) = (1/8)(1/7)$. So we conclude that $\mathrm{Var}(X_1+\cdots+X_8) = ((8)(1/8) + (56)(1/8)(1/7)) - (1)^2 = 1 + 1 - 1 = 1$.

2a. We compute $P(Y = X) = \sum_{x=1}^{5}(1/5)(2/3)^{x-1}(1/3) = (1/5)(1/3)(1 - (2/3)^5)/(1 - 2/3) = (1/5)(1 - (2/3)^5) = 211/1215 = 0.1737$.
2b. We compute $P(Y > X) = \sum_{x=1}^{5}(1/5)(2/3)^x = (1/5)(2/3 - (2/3)^6)/(1 - 2/3) = 422/1215 = 0.3473$.

3a. We compute that $P(Y = X) = \sum_{x=1}^{\infty} \frac{e^{-5}5^x}{x!}(2/3)^{x-1}(1/3) = (e^{-5})(\frac{1/3}{2/3}) \sum_{x=1}^{\infty} \frac{5^x}{x!}(2/3)^x = (e^{-5})(1/2) \sum_{x=1}^{\infty} \frac{(10/3)^x}{x!} = (e^{-5})(1/2)(e^{10/3} - 1) = 0.0911$.
3b. We get $P(Y > X) = \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} \frac{e^{-5}5^x}{x!}(2/3)^{y-1}(1/3) = \sum_{x=0}^{\infty} \frac{e^{-5}5^x}{x!}(1/3) \sum_{y=x+1}^{\infty}(2/3)^{y-1} = \sum_{x=0}^{\infty} \frac{e^{-5}5^x}{x!}(1/3)(2/3)^x/(1 - 2/3) = \sum_{x=0}^{\infty} \frac{e^{-5}5^x}{x!}(2/3)^x = \sum_{x=0}^{\infty} \frac{e^{-5}(10/3)^x}{x!} = (e^{-5})(e^{10/3}) = e^{-5/3} = 0.1889$.

4. In all cases below, we let $X_j = 1$ if the jth person is happy, and $X_j = 0$ otherwise. We let X denote the total number of happy students. So we have $X = X_1 + \cdots + X_7$.
4a. We have $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_7) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_7) = 1/7 + \cdots + 1/7 = 1$.
4b. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. We compute $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_7)^2) = 7\mathbb{E}(X_1^2) + 42\mathbb{E}(X_1X_2) = 7\mathbb{E}(X_1)+42\mathbb{E}(X_1X_2) = 7(1/7)+42(0) = 1$. So we conclude that $\mathrm{Var}(X) = 1-1^2 = 0$.

Note: A different way to derive the answer to **4a** and **4b** is to note that exactly 1 student will always get the blue pencil, so there is always exactly 1 happy student, so the expected number of happy students is 1 and the variance of the number of happy students is 0 (because the random variable is just a *constant* 1 (always) in that setup).
4c. We have $\mathbb{E}(X) = 1$ for the exact same reasoning as in **4a**.
4d. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. We compute $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_7)^2) = 7\mathbb{E}(X_1^2) + 42\mathbb{E}(X_1X_2) = 7\mathbb{E}(X_1) + 42\mathbb{E}(X_1X_2) = 7(1/7) + 42(1/7)(1/6) = 2$. So we conclude that $\mathrm{Var}(X) = 2 - 1^2 = 1$.