STAT/MA 41600
In-Class Problem Set #11: September 14, 2018
Solutions by Mark Daniel Ward

Problem Set 11 Answers

1. Let $X_j = 1$ if the jth card is a Jack, Queen, or King. Let $X_j = 0$ otherwise. Then $\mathbb{E}(X_j) = (1)P(X_j = 1) + (0)P(X_j = 0) = P(X_j = 1) = 12/52 = 3/13$. So we obtain $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_5) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_5) = 3/13 + \cdots + 3/13 = (5)(3/13) = 15/13$.

2. Even though the cards are dealt *without replacement* in this question, and even though the probability distribution is different, we get the same expected value as above, using the same reasoning. This is beautiful and wonderful but also mysterious when students see it. Dr. Ward encourages students to compare with questions and solutions in previous years. This phenomenon is important to understand. We have $\mathbb{E}(X) = 15/13$ for the same reasons.

3. Let $X_j = 1$ if the minimum is bigger than or equal to j, i.e., if $X \geq j$. Let $X_j = 0$ otherwise. Then we have $\mathbb{E}(X_j) = (1)P(X_j = 1) + (0)P(X_j = 0) = P(X_j = 1) = P(X \geq j)$.

So we get $\mathbb{E}(X_1) = 1$, and $\mathbb{E}(X_2) = (3/4)^3$, and $\mathbb{E}(X_3) = (2/4)^3$, and $\mathbb{E}(X_4) = (1/4)^3$. So we obtain $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3 + X_4) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) = 1 + (3/4)^3 + (2/4)^3 + (1/4)^3 = 25/16$.

4. There are lots of ways to set this one up.

Method #1: Let $X_j = 1$ if the jth bear pair is sitting together, and $X_j = 0$ otherwise. Since there are $\binom{6}{2} = 15$ places where each bear pair can sit, and they are only happy in 5 of these places, then $\mathbb{E}(X_j) = 5/15 = 1/3$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 1/3 + 1/3 + 1/3 = 1$.

Method #2: Same definition of X_j, but a different way to find $\mathbb{E}(X_j)$. The father bear sits on the left or right end of the row with probability 2/6, and then the mother bear sits next to him with probability 1/5; or the father bear sits in a middle seat (not at the end of the row) with probability 4/6, and then the mother bear sits next to him with probability 2/5, so $\mathbb{E}(X_j) = (2/6)(1/5) + (4/6)(2/5) = 1/3$, and again we get $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 1/3 + 1/3 + 1/3 = 1$.

Method #3: Let $X_j = 1$ if the jth and $(j+1)$st seats contain a happy bear pair, and $X_j = 0$ otherwise. Regardless of which bear sits in the leftmost of these two seats, the probability that her/his partner is on her/his right is 1/5. So we get $\mathbb{E}(X_j) = 1/5$, so $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3 + X_4 + X_5) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) + \mathbb{E}(X_5) = 1/5 + 1/5 + 1/5 + 1/5 + 1/5 = 1$.