STAT/MA 41600
In-Class Problem Set #9: September 10, 2018
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Problem Set 9 Answers

1. The probability that Y is greater than or equal to X is $\sum_{x=1}^{\infty}\sum_{y=x}^{\infty}(3/4)^{x-1}(1/4)(48/52)^{y-1}(4/52) = \sum_{x=1}^{\infty}(3/4)^{x-1}(1/4)(4/52)\sum_{y=x}^{\infty}(48/52)^{y-1}$, and we see that the inner sum is $\sum_{y=x}^{\infty}(48/52)^{y-1} = (48/52)^{x-1}/(1-48/52)$.

So the desired probability becomes $\sum_{x=1}^{\infty}(3/4)^{x-1}(1/4)(4/52)(48/52)^{x-1}/(1-48/52) = \sum_{x=1}^{\infty}(3/4)^{x-1}(1/4)(48/52)^{x-1} = \sum_{x=1}^{\infty}(1/4)(36/52)^{x-1} = (1/4)/(1-36/52) = 13/16$.

2. The probability is $(1/4)(0)+(1/4)(1/6)(1/8)+(1/4)(2/6)(2/8)+(1/4)(3/6)(3/8) = 7/96$.

3. We look at the 24 possible outcomes, and we compute $p_X(0) = 4/24$, $p_X(1) = 7/24$, $p_X(2) = 6/24$, $p_X(3) = 4/24$, $p_X(4) = 2/24$, and $p_X(5) = 1/24$.

4a. We compute $P(X > 4 \mid Y = 1) = \frac{P(X>4\ \&\ Y=1)}{P(X\geq 1\ \&\ Y=1)} = \frac{\sum_{x=5}^{\infty}(11/16)(1/4)^{x-1}(1/3)^{1-1}}{\sum_{x=1}^{\infty}(11/16)(1/4)^{x-1}(1/3)^{1-1}} = \frac{\sum_{x=5}^{\infty}(1/4)^{x-1}}{\sum_{x=1}^{\infty}(1/4)^{x-1}} = \frac{(1/4)^4/(1-1/4)}{1/(1-1/4)} = (1/4)^4 = 1/256$.

4b. The random variables X and Y are dependent because we need to have $Y \geq X$ in this setup.

4c. For positive integers $y \geq 1$, we compute $p_Y(y) = \sum_{x=y}^{\infty}(11/16)(1/4)^{x-1}(1/3)^{y-1} = (11/16)(1/3)^{y-1}\sum_{x=y}^{\infty}(1/4)^{x-1} = (11/16)(1/3)^{y-1}(1/4)^{y-1}/(1-1/4) = (11/12)(1/12)^{y-1}$, and $p_Y(y) = 0$ otherwise.

Here is another method for question 1, which does not require a double sum: On any given flip, Bob gets blue and Cynthia gets an Ace with probability $(1/4)(4/52)$ (this implies $X = Y$); or Bob gets blue and Cynthia does not get an Ace with probability $(1/4)(48/52)$ (this implies $Y > X$); or Bob gets white and Cynthia gets an Ace with probability $(3/4)(4/52)$ (this implies $X > Y$). If Bob gets white and Cynthia does not get an Ace, they just proceed to another round. So we conclude that $P(Y \geq X) = \frac{(1/4)(4/52)+(1/4)(48/52)}{(1/4)(4/52)+(1/4)(48/52)+(3/4)(4/52)} = 13/16$.