STAT/MA 41600
Midterm Exam 1 Answers
Friday, October 6, 2017
Solutions by Mark Daniel Ward

1. Let A_j be the event that the jth person gets the proper coat. The probability *at least one* person gets their correct coat is $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = 1/3 + 1/3 + 1/3 - (1/3)(1/2) - (1/3)(1/2) - (1/3)(1/2) + (1/3)(1/2)(1)$, which we can simplify to $P(A_1 \cup A_2 \cup A_3) = 1 - 1/2 + 1/6 = 2/3$. So the desired probability is $1 - P(A_1 \cup A_2 \cup A_3) = 1 - 2/3 = 1/3$.

Alternatively (without inclusion-exclusion), for none of them to get their correct coat, the first person just needs to get someone else's coat, which happens with probability $2/3$, and then (afterwards) that person needs to get the remaining person's coat, which happens with probability $1/2$. So the probability none of them get their correct coat is $(2/3)(1/2) = 1/3$.

2a. The X and Y are independent Geometric random variables, with parameters $p = 1/3$ and $p = 3/4$, respectively.

2b. Since X and Y are independent, we can add the variances as follows: $\mathrm{Var}\,(X - Y) = \mathrm{Var}(X) + (-1)^2\,\mathrm{Var}(Y) = \frac{2/3}{(1/3)^2} + \frac{1/4}{(3/4)^2} = 58/9 = 6.4444$.

2c. We compute $P(X = Y) = \sum_{n=1}^{\infty} p_{X,Y}(n,n) = \sum_{n=1}^{\infty} (1/3)(2/3)^{n-1}(3/4)(1/4)^{n-1} = (1/3)(3/4)\sum_{n=1}^{\infty}(1/6)^{n-1} = (1/4)/(1 - 1/6) = (1/4)/(5/6) = 3/10$.

3a. We can write $X = X_1 + \cdots + X_{10}$ where $X_j = 1$ if the jth pair has 1 red and 1 green, or $X_j = 0$ otherwise. Then $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10})$. Also $\mathbb{E}(X_j) = 10/19$, so it follows that $\mathbb{E}(X) = (10)(10/19) = 100/19 = 5.2632$.

3b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{10})^2)$, which has 10 terms of the form $\mathbb{E}(X_j^2)$ and $10^2 - 10 = 90$ terms of the form $\mathbb{E}(X_i X_j)$. Also $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 10/19$ and $\mathbb{E}(X_i X_j) = (10/19)(9/17) = 90/323$. Thus $\mathbb{E}(X^2) = (10)(10/19) + (90)(90/323) = 9800/323 = 30.3406$. So altogether we have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 30.3406 - (5.2632)^2 = 2.64$.

4. We get $P(X = 0) = \binom{3}{0}\binom{3}{2}/\binom{6}{2} = 1/5$, $P(X = 1) = \binom{3}{1}\binom{3}{1}/\binom{6}{2} = 3/5$, and $P(X = 2) = \binom{3}{2}\binom{3}{0}/\binom{6}{2} = 1/5$, so $P(X = Y) = P(X = Y = 0) + P(X = Y = 1) + P(X = Y = 2) = (1/5)^2 + (3/5)^2 + (1/5)^2 = 11/25$.

5abc. We calculate:

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{P(A)}{P(D)} = \frac{\binom{5}{1}P(1,1,1,1,4)}{\binom{5}{1}P(1,1,1,1,4) + \binom{5}{1}\binom{4}{1}P(1,1,1,2,3) + \binom{5}{3}P(1,1,2,2,2)} = \frac{\binom{5}{1}}{\binom{5}{1} + \binom{5}{1}\binom{4}{1} + \binom{5}{3}} = 1/7.$$

$$P(B \mid D) = \frac{P(B \cap D)}{P(D)} = \frac{P(B)}{P(D)} = \frac{\binom{5}{1}\binom{4}{1}P(1,1,1,2,3)}{\binom{5}{1}P(1,1,1,1,4) + \binom{5}{1}\binom{4}{1}P(1,1,1,2,3) + \binom{5}{3}P(1,1,2,2,2)} = \frac{\binom{5}{1}\binom{4}{1}}{\binom{5}{1} + \binom{5}{1}\binom{4}{1} + \binom{5}{3}} = 4/7.$$

$$P(C \mid D) = \frac{P(C \cap D)}{P(D)} = \frac{P(C)}{P(D)} = \frac{\binom{5}{3}P(1,1,2,2,2)}{\binom{5}{1}P(1,1,1,1,4) + \binom{5}{1}\binom{4}{1}P(1,1,1,2,3) + \binom{5}{3}P(1,1,2,2,2)} = \frac{\binom{5}{3}}{\binom{5}{1} + \binom{5}{1}\binom{4}{1} + \binom{5}{3}} = 2/7.$$

Question 1 was the very same as question 4a in problem set 4 from 2014.

Question 2a is to help realize that these are Geometric random variables. Question 2b is like question 3b in problem set 16 from 2016. Question 2c is the very same as question 3a in problem set 9 from 2016.

Question 3 was the very same as question 3 in problem set 22 from 2015, on sums of indicator random variables.

Question 4b was the very same as question 2 in problem set 19 from 2015. (Question 4a just points out the need to first find the probability mass function.)

Question 5 was the very same as question 2 in problem set 4 from 2017.