STAT/MA 41600
In-Class Problem Set #44: December 8, 2017
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Problem Set 44 Answers

1a. Since the probability that a point is in a region is proportion to the area A of that region, it must be the case that the probability is $A/(3\pi^2)$. So the probability of being within a circle of radius 1/5 is $(\pi/5^2)/(3^2\pi) = 1/225$.
1b. The probability is $((7/3)^2\pi)/(3^2\pi) = 49/81$.
1c. The probability is $(a^2\pi)/(3^2\pi) = a^2/9$.
1d. Same reasoning: The probability is exactly the same as in part c, namely, $F_X(a) = P(X \leq a) = a^2/9$.
1e. The density is $f_X(x) = \frac{d}{dx}(x^2/9) = 2x/9$ for $0 \leq x \leq 3$, and $f_X(x) = 0$ otherwise.
1f. We verify that $2x/9 \geq 0$ for $0 \leq x \leq 3$, and also $\int_0^3 2x/9\,dx = x^2/9|_{x=0}^3 = 1$.

2a. We see that, for $a > 0$, we have $F_Y(a) = P(Y \leq a) = P(5X \leq a) = P(X \leq a/5) = 1 - e^{-\lambda a/5}$, and $F_Y(a) = 0$ otherwise, so Y is an exponential random variable with parameter $\lambda/5$. Thus, we get $\mathbb{E}(Y) = 5/\lambda$.
2b. Similar to the previous part, for $a > 0$, we have $F_W(a) = P(W \leq a) = P(W/60 \leq a/60) = P(V \leq a/60) = 1 - e^{-(1/3)a/60} = 1 - e^{-a/180}$, and $F_W(a) = 0$ otherwise, so W is an exponential random variable with parameter $\lambda = 1/180$. Thus, we get $\mathbb{E}(W) = 180$. (This makes sense, since the expected waiting time is 3 minutes which equals 180 seconds.)

3a. We see that Y cannot be a Binomial random variable, because Y only takes on possible values $0, 3, 6, 9, 12, 15$, rather than being able to take on the full range of values $0, \ldots, 15$.
3b. Since X is a Binomial random variable with $n = 5$ and $p = 1/7$, then $\mathbb{E}(X) = np = 5/7$ and $\mathrm{Var}(X) = np(1-p) = (5)(1/7)(6/7) = 30/49$. Therefore, we get $\mathbb{E}(Y) = \mathbb{E}(3X) = 3\mathbb{E}(X) = (3)(5/7) = 15/7$ and $\mathrm{Var}(Y) = \mathrm{Var}(3X) = 9\,\mathrm{Var}(X) = (9)(30/49) = 270/49$.

4a. The random variable X cannot be a Binomial random variable. If it was going to be a Binomial random variable, it would take on possible values $0, \ldots, 7$, but we know that X cannot take on the value 6, because it is impossible for exactly 6 family members to get a matching color iPhone and protective case. Therefore, we conclude that X is not a Binomial random variable.
4b. Let $X_j = 1$ if the jth family member gets a matching color iPhone and protective case. Then we have $X = X_1 + \cdots + X_7$, so we get $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_7) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_7) = 1/7 + \cdots + 1/7 = (7)(1/7) = 1$. We also have $\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathrm{Cov}(X_1 + \cdots + X_7, X_1 + \cdots + X_7) = 7\,\mathrm{Cov}(X_1, X_1) + 42\,\mathrm{Cov}(X_1, X_2) = 7(\mathbb{E}(X_1) - \mathbb{E}(X_1^2)) + 42(\mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)) = 7(1/7 - (1/7)^2) + 42((1/7)(1/6) - (1/7)^2) = 1$.

[[Notice that, if you *incorrectly guessed* that X was Binomial with $n = 7$ and $p = 1/7$ (which is not correct), then you would get expected value $np = (7)(1/7) = 1$ (which is correct) and variance $np(1-p) = (7)(1/7)(6/7) = 6/7 \neq 1$, so you would get the correct expected value but the incorrect variance.]]