STAT/MA 41600
In-Class Problem Set #42: December 1, 2017
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Problem Set 42 Answers

1a. The density of $X_{(1)}$ is $f_{X_{(1)}}(x) = \binom{3}{0,1,2}(x^2/72)(x^3/216)^0(1-x^3/216)^2 = x^2/24 - x^5/2592 + x^8/1119744$ for $0 < x < 6$, and $f_{X_{(1)}}(x) = 0$ otherwise.
1b. The expected value is $\mathbb{E}(X_{(1)}) = \int_0^6 (x)(x^2/24 - x^5/2592 + x^8/1119744)\,dx = (x^4/96 - x^7/18144 + x^{10}/11197440)|_{x=0}^6 = 243/70 = 3.4714$.
1c. The density of $X_{(2)}$ is $f_{X_{(2)}}(x) = \binom{3}{1,1,1}(x^2/72)(x^3/216)^1(1-x^3/216)^1 = x^5/2592 - x^8/559872$ for $0 < x < 6$, and $f_{X_{(2)}}(x) = 0$ otherwise.
1d. The expected value is
$\mathbb{E}(X_{(2)}) = \int_0^6 (x)(x^5/2592 - x^8/559872)\,dx = (x^7/18144 - x^{10}/5598720)|_{x=0}^6 = 162/35 = 4.6286$.

2a. The density of $X_{(3)}$ is $f_{X_{(3)}}(x) = \binom{3}{2,1,0}(x^2/72)(x^3/216)^2(1-x^3/216)^0 = x^8/1119744$ for $0 < x < 6$, and $f_{X_{(3)}}(x) = 0$ otherwise.
2b. The expected value is $\mathbb{E}(X_{(3)}) = \int_0^6 (x)(x^8/1119744)\,dx = x^{10}/11197440|_{x=0}^6 = 27/5 = 5.4$.
2c. We verify that $\mathbb{E}(X_{(1)}) + \mathbb{E}(X_{(2)}) + \mathbb{E}(X_{(3)}) = 243/70 + 162/35 + 27/5 = 27/2$.

3. The density of $X_{(2)}$ is $f_{X_{(2)}}(x) = \binom{3}{1,1,1}(1/10)(x/10)^1(1-x/10)^1 = x/100 - x^2/1000$ for $0 < x < 10$, and $f_{X_{(2)}}(x) = 0$ otherwise.

The expected value is $\mathbb{E}(X_{(2)}) = \int_0^{10} (x)(x/100 - x^2/1000)\,dx = (x^3/50 - 3x^4/2000)|_{x=0}^{10} = 5$.

4. Method #1: The density of $X_{(1)}$ is $f_{X_{(1)}}(x) = \binom{3}{0,1,2}(\frac{1}{5}e^{-x/5})(1-e^{-x/5})^0(e^{-x/5})^2 = \frac{3}{5}e^{-3x/5}$ for $x > 0$, and $f_{X_{(1)}}(x) = 0$ otherwise. Therefore, $f_{X_{(1)}}(x)$ has the form of the density of an Exponential random variable with parameter $\lambda = 3/5$, so the minimum of these three given random variables must be an Exponential random variable with parameter $\lambda = 3/5$.

Method #2: We know that the minimum of independent Exponential random variables is an Exponential random variable too (this is not true for the maximum or for the other order statistics of Exponential random variables!). Moreover, we add the parameters of these Exponential random variables to get the parameter of the minimum. In this case, the minimum is an Exponential random variable with parameter $\lambda = 1/5 + 1/5 + 1/5 = 3/5$.

Using either method to discover that the minimum is an Exponential random variable with parameter $\lambda = 3/5$, we can conclude that the expected value and variance of the minimum are $1/\lambda = 5/3$ and $1/\lambda^2 = 25/9$, respectively.