STAT/MA 41600
In-Class Problem Set #34: November 3, 2017
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Problem Set 34 Answers

1. We have $P(|X-1/2|<1/6) = \int_{1/3}^{2/3} f_X(x)\,dx = \int_{1/3}^{2/3} \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx = \int_{1/3}^{2/3} 12x^2(1-x)\,dx = \int_{1/3}^{2/3} 12(x^2-x^3)\,dx = 12(x^3/3 - x^4/4)\big|_{x=1/3}^{2/3} = 16/27 - 1/9 = 13/27.$

2. We have $P(X<0.3) = \int_0^{0.3} f_X(x)\,dx = \int_0^{0.3} \frac{\Gamma(3+4)}{\Gamma(3)\Gamma(4)} x^{3-1}(1-x)^{4-1}\,dx = \int_0^{0.3} 60x^2(1-x)^3\,dx = \int_0^{0.3} 60(x^2-3x^3+3x^4-x^5)\,dx = 60(x^3/3 - 3x^4/4 + 3x^5/5 - x^6/6)\big|_0^{0.3} = .2557.$

3. We have $P(U>X) = P(1<U<3) + P(X<U<1) = 2/3 + \int_0^1 \int_0^u (1/3)(12(x^2-x^3))\,dx\,du = 2/3 + \int_0^1 (1/3)(12(x^3/3 - x^4/4))\big|_{x=0}^{u}\,du = 2/3 + \int_0^1 (1/3)(12(u^3/3 - u^4/4))\,du = 2/3 + (1/3)(12(u^4/12 - u^5/20))\big|_{u=0}^{1} = 2/3 + (1/3)(12(1/12 - 1/20)) = 4/5.$

4. We compute

$$\begin{aligned}
P(Y<X) &= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty} (2/3)^{x-1}(1/3)(1/5)^{y-1}(4/5) \\
&= \sum_{y=1}^{\infty} (1/3)(1/5)^{y-1}(4/5)\sum_{x=y}^{\infty}(2/3)^x \\
&= \sum_{y=1}^{\infty} (1/3)(1/5)^{y-1}(4/5)\frac{(2/3)^y}{1-2/3} \\
&= \sum_{y=1}^{\infty} (2/15)^{y-1}(4/5)(2/3) \\
&= \sum_{y=0}^{\infty} (2/15)^{y}(4/5)(2/3) \\
&= \frac{1}{1-2/15}(4/5)(2/3) \\
&= 8/13
\end{aligned}$$