STAT/MA 41600
In-Class Problem Set #29: October 23, 2017
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Problem Set 29 Answers

1. We have $\mathbb{E}(X^2) = \int_0^3 (x^2)(x/9)\,dx + \int_3^6 (x^2)(2/3 - x/9)\,dx = x^4/36|_{x=0}^3 + (2x^3/9 - x^4/36)|_{x=3}^6 = 9/4 + (48-36) - (6-9/4) = 21/2$. So we conclude that $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 21/2 - 3^2 = 3/2$.

2a. We compute $\mathbb{E}(X^2) = \int_0^2 \int_0^x (x^2)(3/4)(x-y)\,dy\,dx = \int_0^2 (x^2)(3/4)(xy - y^2/2)|_{y=0}^x\,dx = \int_0^2 (3/4)(x^4/2)\,dx = (3/4)(x^5/10)|_{x=0}^2 = (3/4)(16/5) = 12/5$. Therefore, we get $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 12/5 - (3/2)^2 = 3/20$.
2b. We compute $\mathbb{E}(XY) = \int_0^2 \int_0^x (xy)(3/4)(x-y)\,dy\,dx = \int_0^2 (3/4)(x^2y^2/2 - xy^3/3)|_{y=0}^x\,dx = \int_0^2 (3/4)(x^4/6)\,dx = (3/4)(x^5/30)|_{x=0}^2 = (3/4)(32/30) = 4/5$.

3. We compute that $\mathbb{E}(Y^2) = \int_0^\infty \int_{5y}^\infty (y^2)(69e^{-3x-8y})\,dx\,dy = \int_0^\infty (y^2)(-23e^{-3x-8y})|_{x=5y}^\infty\,dy = \int_0^\infty (y^2)(23e^{-23y})\,dy = (y^2)(-e^{-23y})|_{y=0}^\infty - \int_0^\infty (-2ye^{-23y})\,dy = \int_0^\infty (2ye^{-23y})\,dy$, and then we use partial fractions again to get $\int_0^\infty (2ye^{-23y})\,dy = (2y)(-e^{-23y}/23)|_{y=0}^\infty - \int_0^\infty (-2e^{-23y}/23)\,dy = -2e^{-23y}/(23)^2|_{y=0}^\infty = 2/(23)^2$. So we conclude that $\text{Var}(Y) = 2/(23)^2 - (1/23)^2 = 1/23^2 = 1/529$.

4a. We have $\mathbb{E}(XY) = \int_0^2 \int_{2y-4}^{-4y+8} (xy)(1/12)\,dx\,dy = \int_0^2 (x^2/2)(y)(1/12)|_{x=2y-4}^{-4y+8}\,dy = \int_0^2 ((-4y+8)^2/2 - (2y-4)^2/2)(y)(1/12)\,dy = \int_0^2 (6y^3 - 24y^2 + 24y)(1/12)\,dy = (6y^4/4 - 8y^3 + 12y^2)(1/12)|_{y=0}^2 = (24 - 64 + 48)(1/12) = 2/3$.
4b. We have $\mathbb{E}(Y^2) = \int_0^2 \int_{2y-4}^{-4y+8} (y^2)(1/12)\,dx\,dy = \int_0^2 (xy^2)(1/12)|_{x=2y-4}^{-4y+8}\,dy = \int_0^2 (y^2)((-4y+8) - (2y-4))(1/12)\,dy = \int_0^2 (y^2)(-6y+12)(1/12)\,dy = \int_0^2 (-6y^3 + 12y^2)(1/12)\,dy = (-6y^4/4 + 4y^3)(1/12)|_{y=0}^2 = (-24+32)(1/12) = 8/12 = 2/3$. So we conclude that $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/3 - (2/3)^2 = 6/9 - 4/9 = 2/9$.