STAT/MA 41600
In-Class Problem Set #20/#22: October 2, 2017
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Problem Set 20/22 Answers

1. Let $X_j = 1$ if the jth couple sits together, and $X_j = 0$ otherwise. Then we get $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_5) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_5) = 9/\binom{10}{2} + \cdots + 9/\binom{10}{2} = 1$.

Alternatively, we let $Y_j = 1$ if the jth and $(j+1)$st chairs have a couple sitting in this pair of chairs. We get $\mathbb{E}(X) = \mathbb{E}(Y_1 + \cdots + Y_9) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_9) = 1/9 + \cdots + 1/9 = 1$.

2. The probability that the first two chairs contain a couple is $1/9$. Given that the first two chairs contain a couple, the conditional probability that the next two chairs contain a couple is $1/7$. Given that the first four chairs contain two couples, the conditional probability that the next two chairs contain a couple is $1/5$. Given that the first six chairs contain two couples, the conditional probability that the next two chairs contain a couple is $1/3$. Finally, that means that the last two chairs contain a couple too. So the desired probability is $(1/9)(1/7)(1/5)(1/3) = 1/945$.

3abc. There are $\binom{40}{4}$ equally likely ways to choose who is a member of Group A, and exactly $\binom{39}{3}$ of these ways have Bob as a member. So the desired probability is $\binom{39}{3}/\binom{40}{4} = 1/10$. An alternative approach for **3c** is to notice that Bob is equally likely to be in any of the 10 groups, so that desired probability is $1/10$.

4a. We let $X_j = 1$ if the jth student gets two bears of the same color, and $X_j = 0$ otherwise. So we get $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10}) = 9/19 + \cdots + 9/19 = 90/19$.

4b. We let $Y_j = 1$ if the jth student gets two bears of different colors, and $Y_j = 0$ otherwise. So we get $\mathbb{E}(Y) = \mathbb{E}(Y_1 + \cdots + Y_{10}) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_{10}) = 10/19 + \cdots + 10/19 = 100/19$.

Double checking: We have $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 90/19 + 100/19 = 190/19 = 10$.

4c. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. We know that $(\mathbb{E}(X))^2 = (90/19)^2$. We also have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{10})^2) = 10\mathbb{E}(X_1^2) + 90\mathbb{E}(X_1X_2)$. We know $\mathbb{E}(X_1^2) = \mathbb{E}(X_1) = 9/19$. We also have $\mathbb{E}(X_1X_2) = (9/19)((8/18)(7/17) + (10/18)(9/17)) = 73/323$. So altogether we get $\mathrm{Var}(X) = 10(9/19) + 90(73/323) - (90/19)^2 = 16200/6137 = 2.6397$.

An alternative approach is to use $\mathbb{E}(X_1X_2) = (9/19)(\binom{8}{2} + \binom{10}{2})/\binom{18}{2} = 73/323$.