STAT/MA 41600
In-Class Problem Set #15: September 20, 2017
Solutions by Mark Daniel Ward

Problem Set 15 Answers

1. The probability that X is an even integer is $\binom{6}{0}(1/3)^0(2/3)^6+\binom{6}{2}(1/3)^2(2/3)^4+\binom{6}{4}(1/3)^4(2/3)^2+\binom{6}{6}(1/3)^6(2/3)^0 = 365/729$. Similarly, the probability that X is an odd integer is $\binom{6}{1}(1/3)^1(2/3)^5+\binom{6}{3}(1/3)^3(2/3)^3+\binom{6}{5}(1/3)^5(2/3)^1 = 364/729$. So X is slightly more likely to be even.

2a. We have $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = np + np = 21/2$. [Alternatively, since X and Y are independent, we can recognize that $X+Y$ is also a Binomial random variable with parameters $n = 14$ and $p = 3/4$, so $\mathbb{E}(X+Y) = (14)(3/4) = 21/2$.]

Since X and Y are independent, we can add their variances. So we get $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y) = np(1-p) + np(1-p) = 21/8$. [Alternatively, since $X+Y$ is Binomial with parameters $n = 14$ and $p = 3/4$, we have $\mathrm{Var}(X+Y) = (14)(3/4)(1/4) = 21/8$.]

2b. We have $\mathbb{E}(X-Y) = \mathbb{E}(X) - \mathbb{E}(Y) = np - np = 0$.

Since X and Y are independent, then X and $-Y$ are independent too, and we can add their variances: $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(-Y) = \mathrm{Var}(X) + (-1)^2\,\mathrm{Var}(Y) = np(1-p) + np(1-p) = 21/8$.

2c. They cannot have the same distribution, because U can take on *any* integer from 0 to 14, while $2X$ can only take on *even* integer values from 0 to 14.

2d. We have $\mathbb{E}(2X) = 2\mathbb{E}(X) = 2np = 21/2$, so $2X$ has the same expected value as $X+Y$. On the other hand, $\mathrm{Var}(2X) = 2^2\,\mathrm{Var}(X) = (2^2)(n)(p)(1-p) = 21/4$, so $2X$ has a different variance than $X+Y$.

3a. By symmetry, we see that $P(X > Y)$ and $P(X < Y)$ are the same value, and also we know $P(X > Y) + P(X < Y) + P(X = Y) = 1$, and therefore, $2P(X > Y) + P(X = Y) = 1$, so $P(X > Y) = (1 - P(X = Y))/2$. We have:

$P(X = Y) = (\binom{5}{0}(2/3)^0(1/3)^5)^2+(\binom{5}{1}(2/3)^1(1/3)^4)^2+(\binom{5}{2}(2/3)^2(1/3)^3)^2+(\binom{5}{3}(2/3)^3(1/3)^2)^2+(\binom{5}{4}(2/3)^4(1/3)^1)^2+(\binom{5}{5}(2/3)^5(1/3)^0)^2 = 575/2187$.

Therefore, we get $P(X > Y) = (1 - 575/2187)/2 = 806/2187$.

3b. Since X and Y are independent, then X and $-Y$ are independent too, so we can add their variances, namely, $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(-Y) = \mathrm{Var}(X) + (-1)^2\,\mathrm{Var}(Y) = np(1-p) + np(1-p) = 20/9$.

3c. We compute $P(X-Y \geq 3) = P(X = 3\ \&\ Y = 0)+P(X = 4\ \&\ Y = 0)+P(X = 5\ \&\ Y = 0)+P(X = 4\ \&\ Y = 1)+P(X = 5\ \&\ Y = 1)+P(X = 5\ \&\ Y = 2) = \binom{5}{3}(2/3)^3(1/3)^2\binom{5}{0}(2/3)^0(1/3)^5+\binom{5}{4}(2/3)^4(1/3)^1\binom{5}{0}(2/3)^0(1/3)^5+\binom{5}{5}(2/3)^5(1/3)^0\binom{5}{0}(2/3)^0(1/3)^5+\binom{5}{4}(2/3)^4(1/3)^1\binom{5}{1}(2/3)^1(1/3)^4+\binom{5}{5}(2/3)^5(1/3)^0\binom{5}{1}(2/3)^1(1/3)^4+\binom{5}{5}(2/3)^5(1/3)^0\binom{5}{2}(2/3)^2(1/3)^3 = 32/729$.

4a. The number of 3's is a Binomial random variable with $n = 20$ and $p = 1/5$ so the expected number of 3's is $np = 4$.

4b. The number of even rolls is a Binomial random variable with $n = 20$ and $p = 3/5$ so the expected number even rolls is $np = 12$.

4c. The number of even rolls is a Binomial random variable with $n = 20$ and $p = 3/5$ so the variance of the number of even rolls is $np(1-p) = 24/5$.

4d. The number of even rolls is a Binomial random variable with $n = 20$ and $p = 2/5$ so the variance of the number of odd rolls is $np(1-p) = 24/5$. It does not matter whether we include the 21st roll, because if we include the 21st roll, it is simply an extra constant of 1, and constants do not affect the variance, i.e., $\mathrm{Var}(X) = \mathrm{Var}(X+1)$ for all random variables X.