STAT/MA 41600
Midterm Exam 2 Answers
Wednesday, November 16, 2016
Solutions by Mark Daniel Ward

1. We have $P(Y \le 3X) = \int_0^\infty \int_{y/3}^\infty 21e^{-3x-7y}\,dx\,dy = \int_0^\infty 7e^{-8y}\,dy = 7/8$.

2. Let X denote the number of jelly beans produced during the 1 hour. Then $P(41600 \le X \le 41700) = P(41599.5 \le X \le 41700.5) = P\big(\frac{41599.5-41666.67}{\sqrt{41666.67}} \le \frac{X-41666.67}{\sqrt{41666.67}} \le \frac{41700.5-41666.67}{\sqrt{41666.67}}\big) \approx P(-0.33 \le Z \le 0.17) = P(Z \le 0.17) - P(Z \le -0.33) = P(Z \le 0.17) - P(Z \ge 0.33) = P(Z \le 0.17) - (1 - P(Z \le 0.33)) = 0.5675 - (1 - 0.6293) = 0.1968$.

3. Let X be the number of students who attend. Then X is Binomial with $n = 400$, $p = 0.60$, so $P(230 \le X \le 250) = P(229.5 \le X \le 250.5) = P\Big(\frac{229.5-(400)(0.60)}{\sqrt{(400)(0.60)(0.40)}} \le \frac{X-(400)(0.60)}{\sqrt{(400)(0.60)(0.40)}} \le \frac{250.5-(400)(0.60)}{\sqrt{(400)(0.60)(0.40)}}\Big) \approx P(-1.07 \le Z \le 1.07) = P(Z \le 1.07) - P(Z < -1.07) = P(Z \le 1.07) - P(Z > 1.07) = P(Z \le 1.07) - (1 - P(Z \le 1.07)) = 2P(Z \le 1.07) - 1 = 2(.8577) - 1 = .7154$.

4. The probability is $\int_0^{10}\int_x^\infty (\frac{1}{10})(\frac{1}{5})e^{-y/5}\,dy\,dx = \int_0^{10} -\frac{1}{10}e^{-y/5}\big|_{y=x}^\infty\,dx = \int_0^{10} \frac{1}{10}e^{-x/5}\,dx = \frac{1}{2}\int_0^{10} \frac{1}{5}e^{-x/5}\,dx$, but the last integral is just the CDF of an exponential with average of 5, evaluated at 10. So the overall probability is $\frac{1}{2}(1 - e^{-10/5}) = \frac{1}{2}(1 - e^{-2}) = 0.4323$.

5. We have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$. The numerator is $10e^{-3x-2y}$. The denominator is $f_X(x) = \int_x^\infty 10e^{-3x-2y}\,dy = 5e^{-5x}$. Thus $f_{Y|X}(y \mid x) = \frac{10e^{-3x-2y}}{5e^{-5x}} = 2e^{2x-2y}$ for $x < y < \infty$, and $f_{Y|X}(y \mid x) = 0$ otherwise. When $x = 2$, we get $f_{Y|X}(y \mid 2) = 2e^{2(2)-2y} = 2e^{4-2y}$. Thus $P(Y > 3 \mid X = 2) = \int_3^\infty f_{Y|X}(y \mid 2)\,dy = \int_3^\infty 2e^{4-2y}\,dy = e^{-2}$.

Bonus. The probability Z exceeds $X+Y$ is $\int_0^1 P(X+Y \le z)f_Z(z)\,dz = \int_0^1 (z^2/2)(1)\,dz = 1/6$. The same probability holds for X to exceed $Y+Z$ or for Y to exceed $X+Z$. So the desired probability is $(3)(1/6) = 1/2$.

Question 1 was the same as question 3b in problem set 25 from 2015.
Question 2 was like question 4b in problem set 37 from 2016, with the numbers changed.
Question 3 was the same as question 2 in "practice problems", problem set 37, part 2.
Question 4 was the same as question 5 in "practice problems", problem set 32.
Question 5 was the same as question 3b in problem set 27 from 2014.