STAT/MA 41600
Midterm Exam 1 Answers
Friday, October 7, 2016
Solutions by Mark Daniel Ward

1. Let S be the event that the student is from Science, and let L be the event that the student liked the lecture. Then $P(S \mid L) = \frac{P(S\cap L)}{P(L)} = \frac{(.15)(.90)}{(.15)(.90)+(.21)(.18)+(.24)(0)+(.40)(.10)} = 0.6344$.

2a. Since X and Y are independent Geometric random variables, each with $p = 1/6$, then $\mathrm{Var}\,(X - Y) = \mathrm{Var}(X) + (-1)^2\,\mathrm{Var}(Y) = \frac{5/6}{(1/6)^2} + \frac{5/6}{(1/6)^2} = 30 + 30 = 60$.
2b. We compute $P(X = Y) = \sum_{n=1}^{\infty} P(X = Y = n) = \sum_{n=1}^{\infty}(5/6)^{n-1}(1/6)(5/6)^{n-1}(1/6) = (1/36)\sum_{n=1}^{\infty}(25/36)^{n-1} = (1/36)(\frac{1}{1-25/36}) = 1/11$.

3a. The number X of Deluxe harmonicas is Hypergeometric with $M = 7$, $N = 19$, and $n = 8$. So $\mathbb{E}(X) = (8)(7/19) = 56/19 = 2.9474$. Alternatively, this can be seen by the fact that each harmonica selected has a probability 7/19 of being a Deluxe harmonica.
3b. Since the number X of Deluxe harmonicas is Hypergeometric with $M = 7$, $N = 19$, and $n = 8$, then $P(X = 5) = \frac{\binom{7}{5}\binom{12}{3}}{\binom{19}{8}} = 770/12597 = 0.0611$.
3c. In this case, the number X of Deluxe harmonicas is Binomial with $n = 8$ and $p = 7/19$. So $\mathbb{E}(X) = (8)(7/19) = 56/19 = 2.9474$. Again, this can also be seen by the fact that each harmonica selected has a probability 7/19 of being a Deluxe harmonica.

4. Since X is Negative Binomial with $r = 5$ and $p = 1/3$, then $P(X > 7 \mid X > 5) = \frac{P(X>7\&X>5)}{P(X>5)} = \frac{P(X>7)}{P(X>5)} = \frac{1-P(X\leq 7)}{1-P(X\leq 5)} = \frac{1-P(X=7)-P(X=6)-P(X=5)}{1-P(X=5)} = \frac{1-\binom{6}{4}q^2p^5-\binom{5}{4}qp^5-\binom{4}{4}p^5}{1-\binom{4}{4}p^5} = \frac{116}{121}$.

5a. We have $p_Y(y) = \sum_{x=y}^{\infty}(\frac{2}{3})^x(\frac{3}{8})^y = (\frac{3}{8})^y\sum_{x=y}^{\infty}(\frac{2}{3})^x = (\frac{3}{8})^y(\frac{2}{3})^y/(1-\frac{2}{3}) = (3)(\frac{1}{4})^y = (\frac{3}{4})(\frac{1}{4})^{y-1}$ for $y \geq 1$, and $p_Y(y) = 0$ otherwise.
5b. We note that Y is a geometric random variable with parameter $p = 3/4$.

Bonus. If $n = 1$, then the 1 family is always happy, so the variance is 0. Now consider $n \geq 2$. Let X_j denote whether the jth family is happy, i.e., $X_j = 1$ if the jth family is happy, and $X_j = 0$ otherwise. We compute $\mathbb{E}(X_1 + \cdots + X_n) = n\mathbb{E}(X_1) = (n)(3)(\frac{2}{3n-1})(\frac{1}{3n-2}) = \frac{6n}{(3n-1)(3n-2)}$, and $\mathbb{E}((X_1 + \cdots + X_n)^2) = n\mathbb{E}(X_1) + (n^2 - n)\mathbb{E}(X_1X_2) = \frac{6n}{(3n-1)(3n-2)} + (n^2 - n)\frac{6}{(3n-1)(3n-2)}\frac{3n-5}{\binom{3n-3}{3}} = \frac{6n}{(3n-4)(3n-1)}$, so we conclude $\mathrm{Var}(X_1 + \cdots + X_n) = \frac{6n}{(3n-4)(3n-1)} - (\frac{6n}{(3n-1)(3n-2)})^2$.

Question 1 was like question 2 in problem set 5 from 2015, with only the words changed (but the same numbers).

Question 2 was like questions 3b and 1b in problem set 16 from 2016, with the names of people changed, and the numbers simplified to be a little easier to compute.

Question 3 was like question 2a, 2c, and a comparison to a Binomial, from practice problem set 19.

Question 4 was the very same as question 4a in problem set 17 from 2014.

Question 5 was like question 6c in problem set 9 from 2014, with a small change to the numbers.

The bonus question is motivated by several questions from this semester, for instance, question 4 from problem set 4, and question 1b from problem set 20/22.