STAT/MA 41600
In-Class Problem Set #40: November 28, 2016
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Problem Set 40 Answers

1a. For $x \geq 1$, we compute $p_X(x) = \sum_{y=x}^{\infty}(3/2)(1/2)^x(1/2)^y = (3/2)(1/2)^x \sum_{y=x}^{\infty}(1/2)^y = (3/2)(1/2)^x \frac{(1/2)^x}{1-1/2} = (3)(1/4)^x$.
1b. For $y \geq x \geq 1$, we have $p_{Y|X}(y \mid x) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{(3/2)(1/2)^x(1/2)^y}{(3)(1/4)^x} = (1/2)(1/2)^{y-x}$.
1c. We note that $p_{Y|X}(y \mid x)$ is nonnegative, and we also verify that the conditional probability mass function sums to 1: $\sum_{y=x}^{\infty} p_{Y|X}(y \mid x) = \sum_{y=x}^{\infty}(1/2)(1/2)^{y-x} = (1/2)\frac{(1/2)^{x-x}}{1-1/2} = 1$.

2a. We have $\mathbb{E}(Y \mid X = x) = \sum_{y=x}^{\infty} y\, p_{Y|X}(y \mid x) = \sum_{y=x}^{\infty}(y)(1/2)(1/2)^{y-x}$, and we can shift the index by x, to obtain $\mathbb{E}(Y \mid X = x) = \sum_{y=0}^{\infty}(y+x)(1/2)(1/2)^y = \sum_{y=0}^{\infty}(y)(1/2)(1/2)^y + \sum_{y=0}^{\infty}(x)(1/2)(1/2)^y$. The first sum is $\sum_{y=0}^{\infty}(y)(1/2)(1/2)^y = \sum_{y=1}^{\infty}(y-1)(1/2)(1/2)^{y-1} = \sum_{y=1}^{\infty}(y-1)(1/2)^y$, and if we view $(1/2)^y$ as the probability mass function of a geometric random variable with parameter $p = 1/2$, this is just the expected value minus 1, i.e., it is $\frac{1}{1/2} - 1 = 2 - 1 = 1$. The second sum is $\sum_{y=0}^{\infty}(x)(1/2)(1/2)^y = (x)(1/2)\sum_{y=0}^{\infty}(1/2)^y = (x)(1/2)\frac{(1/2)^0}{1-1/2} = x$. So altogether we have $\mathbb{E}(Y \mid X = x) = x + 1$.
2b. We have $\mathbb{E}(Y \mid X = 5) = 5 + 1 = 6$.

3a. We have $f_X(x) = \int_{y=0}^{2} f_{X,Y}(x,y)\,dy = \int_{y=0}^{2} \frac{1}{12}(4 - xy)\,dy = 2/3 - x/6$ for $0 < x < 2$, and $f_X(x) = 0$ otherwise.
3b. For fixed $0 < x < 2$ and $0 < y < 2$, we have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(1/12)(4-xy)}{2/3-x/6}$.
3c. We have $\mathbb{E}(Y \mid X = x) = \int_0^2 y\, f_{Y|X}(y \mid x)\,dy = \int_0^2 (y)\frac{(1/12)(4-xy)}{(2/3-x/6)}\,dy = \frac{(1/12)\int_0^2 (y)(4-xy)\,dy}{(2/3-x/6)} = \frac{(1/12)(8-8x/3)}{(2/3-x/6)} = \frac{(2/3)(1-x/3)}{(2/3-x/6)}$.
3d. We compute $\mathbb{E}(Y) = \int_0^2 \mathbb{E}(Y \mid X = x) f_X(x)\,dx = \int_0^2 \frac{(2/3)(1-x/3)}{(2/3-x/6)}(2/3 - x/6)\,dx = \int_0^2 (2/3)(1-x/3)\,dx = 8/9$.

4. For $0 < x < 2$, we have $\mathbb{E}(Y \mid X = x) = \int_0^{2x} y\, f_{Y|X}(y \mid x)\,dy = \int_0^{2x}(y)(y/(2x^2))\,dy = 4x/3$.
[Note: Since $Y < 2X$ in this problem, then we know that $\mathbb{E}(Y \mid X = x)$ cannot be bigger than $2x$, and indeed it turns out to be only $4x/3$.]