STAT/MA 41600
In-Class Problem Set #39: November 18, 2016
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Problem Set 39 Answers

1a. We have $\mathbb{E}(X_iX_j) = (2/5)((2/4)(1/3) + (2/4)(2/3)) = 1/5$ and $\mathbb{E}(X_i) = 2/5$ and $\mathbb{E}(X_j) = 2/5$, so $\mathrm{Cov}(X_i, X_j) = 1/5 - (2/5)^2 = 1/25$.
1b. We have $\mathrm{Var}(X_i) = \mathbb{E}(X_iX_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = \mathbb{E}(X_i) - (\mathbb{E}(X_i))^2 = 2/5 - (2/5)^2 = 6/25$.
1c. We conclude that $\mathrm{Var}(X) = 6(1/25) + 3(6/25) = 24/25$.

2a. We have $\mathbb{E}(X_iX_j) = (36/52)(35/51) = 105/221$ and $\mathbb{E}(X_i) = 36/52 = 9/13$ and $\mathbb{E}(X_j) = 36/52 = 9/13$, so $\mathrm{Cov}(X_i, X_j) = 105/221 - (9/13)^2 = -12/2873$.
2b. We have $\mathrm{Var}(X_i) = \mathbb{E}(X_iX_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = \mathbb{E}(X_i) - (\mathbb{E}(X_i))^2 = 9/13 - (9/13)^2 = 36/169$.
2c. We conclude that $\mathrm{Var}(X) = (6)(-12/2873) + (3)(36/169) = 1764/2873$.

3. We have $\mathbb{E}(XY) = \int_0^2\int_0^2 (xy)(\frac{1}{12}(4-xy))\,dx\,dy = \frac{1}{12}\int_0^2\int_0^2 (4xy - x^2y^2)\,dx\,dy = \frac{1}{12}\int_0^2 (2x^2y - x^3y^2/3)|_{x=0}^2\,dy = \frac{1}{12}\int_0^2 (8y - 8y^2/3)\,dy = \frac{1}{12}(4y^2 - 8y^3/9)|_{y=0}^2 = \frac{1}{12}(16 - 64/9) = 20/27$.

We also have $\mathbb{E}(X) = \int_0^2\int_0^2 (x)(\frac{1}{12}(4-xy))\,dx\,dy = \frac{1}{12}\int_0^2\int_0^2 (4x - x^2y)\,dx\,dy = \frac{1}{12}\int_0^2 (2x^2 - x^3y/3)|_{x=0}^2\,dy = \frac{1}{12}\int_0^2 (8 - 8y/3)\,dy = \frac{1}{12}(8y - 4y^2/3)|_{y=0}^2 = \frac{1}{12}(16 - 16/3) = 8/9$.

Alternatively, from Problem Set 26, question #4b, we know that $f_X(x) = 2/3 - x/6$ for $0 < x < 2$, and $f_X(x) = 0$ otherwise, so $\mathbb{E}(X) = \int_0^2 (x)(2/3 - x/6)\,dx = \int_0^2 (2x/3 - x^2/6)\,dx = (x^2/3 - x^3/18)|_{x=0}^2 = 4/3 - 4/9 = 8/9$.

By symmetry, we know that $\mathbb{E}(Y) = \mathbb{E}(X) = 8/9$. Thus $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 20/27 - (8/9)^2 = -4/81$.

4. For $0 < x < 2$, we have $f_X(x) = \int_0^{2x} \frac{1}{8}xy\,dy = \frac{1}{16}xy^2|_{y=0}^{2x} = \frac{1}{16}x(2x)^2 = \frac{x^3}{4}$. Therefore, the conditional density of Y given $X = x$ is $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(1/8)xy}{x^3/4} = y/(2x^2)$ for $0 < y < 2x$, and $f_{Y|X}(y \mid x) = 0$ otherwise.