STAT/MA 41600
In-Class Problem Set #29: October 24, 2016
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Problem Set 29 Answers

1a. As we noticed before, $f_{X,Y}(x,y)$ can be factored in this problem, so that $f_X(x) = 5e^{-5x}$ for $x > 0$ and $f_X(x) = 0$ otherwise, and $f_Y(y) = 3e^{-3y}$ for $y > 0$ and $f_Y(y) = 0$ otherwise. We already computed $\mathbb{E}(X) = 1/5$. Now we compute $\mathbb{E}(X^2) = \int_0^\infty (x^2)(5e^{-5x})\,dx$. Using integration by parts with $u = x^2$ and $dv = 5e^{-5x}\,dx$, we have $du = 2x\,dx$ and $v = -e^{-5x}$, so $\mathbb{E}(X^2) = (x^2)(-e^{-5x})|_{x=0}^\infty - \int_0^\infty (2x)(-e^{-5x})\,dx = \int_0^\infty 2xe^{-5x}\,dx = (2/5)\int_0^\infty (x)(5e^{-5x})\,dx = (2/5)\mathbb{E}(X) = (2/5)(1/5) = 2/25$.

Alternatively, we could have computed $\mathbb{E}(X^2) = \int_0^\infty\int_0^\infty (x^2)(15e^{-5x-3y})\,dx\,dy = 2/25$.

Finally, we get $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2/25 - (1/5)^2 = 1/25$.

1b. Using the same steps as in 1a, we compute $\mathbb{E}(Y^2) = \int_0^\infty (y^2)(3e^{-3y})\,dy = 2/9$, so $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/9 - (1/3)^2 = 1/9$.

2. We compute $\mathbb{E}(X^2) = \int_0^\infty\int_x^\infty (x^2)(24e^{-5x-3y})\,dy\,dx = \int_0^\infty (x^2)(24e^{-5x})\int_x^\infty (e^{-3y})\,dy\,dx = \int_0^\infty (x^2)(24e^{-5x})(-1/3)(e^{-3y})|_{y=x}^\infty\,dx = \int_0^\infty (x^2)(24e^{-5x})(1/3)(e^{-3x})\,dx = \int_0^\infty (x^2)(8e^{-8x})\,dx$. Now we use integration by parts with $u = x^2$ and $dv = 8e^{-8x}\,dx$, and therefore $du = 2x\,dx$ and $v = -e^{-8x}\,dx$. So we get $\mathbb{E}(X^2) = (x^2)(-e^{-8x})|_{x=0}^\infty - \int_0^\infty (2x)(-e^{-8x})\,dx = \int_0^\infty 2xe^{-8x}\,dx = (1/4)\int_0^\infty 8xe^{-8x}\,dx = (1/4)\mathbb{E}(X) = (1/4)(1/8) = 1/32$. So we have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1/32 - (1/8)^2 = 1/64$.

3a. We compute $\mathbb{E}(X^2) = \int_0^8\int_0^{4-x/2} (x^2)(1/16)\,dy\,dx = \int_0^8 (4 - x/2)(x^2)(1/16)\,dy\,dx = (1/16)\int_0^8 (4x^2 - x^3/2)\,dx = (1/16)(4x^3/3 - x^4/8)|_{x=0}^8 = 32/3$. Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 32/3 - (8/3)^2 = 32/9$.

3b. We compute $\mathbb{E}(Y^2) = \int_0^8\int_0^{4-x/2} (y^2)(1/16)\,dy\,dx = \int_0^8 (y^3/3)(1/16)|_{y=0}^{4-x/2}\,dx = \int_0^8 (4 - x/2)^3(1/48)\,dx = \int_0^8 (64 - 24x + 3x^2 - x^3/8)(1/48)\,dx = (64x - 12x^2 + x^3 - x^4/32)(1/48)|_{x=0}^8 = 8/3$. Thus $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 8/3 - (4/3)^2 = 8/9$.

4. We compute $\mathbb{E}(X^2) = \int_0^2\int_0^2 (x^2)(\frac{1}{12}(4 - xy))\,dy\,dx = \int_0^2 (x^2)(\frac{1}{12}(4y - xy^2/2)|_{y=0}^2)\,dx = \int_0^2 (x^2)(\frac{1}{12}(8 - 2x))\,dx = \int_0^2 \frac{1}{12}(8x^2 - 2x^3)\,dx = \frac{1}{12}(8x^3/3 - x^4/2)|_{x=0}^2 = \frac{1}{12}(64/3 - 8) = 10/9$. Thus $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 10/9 - (8/9)^2 = 26/81$.