STAT/MA 41600
In-Class Problem Set #20/#22: October 3, 2016
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Problem Set 20/22 Answers

1a. The probability that their 2 numbers are adjacent is $299/\binom{300}{2} = 299/(300 \times 299/2) = 2/300 = 1/150$.
1b. The probability that their 3 numbers are adjacent is $48/\binom{50}{3} = 48/(50 \times 49 \times 48/6) = 48/19600 = 3/1225$.

2a. Let X_j indicate if the jth fruit flavored jelly bean is chosen before all of the licorice flavored jelly beans, i.e., $X_j = 1$ if the jth fruit flavored jelly bean is chosen before all of the licorice flavored jelly beans, and $X_j = 0$ otherwise. Then she eats $X_1 + \cdots + X_{30}$ jelly beans.

We have $\mathbb{E}(X_j) = P(X_j = 1) = 1/21$, since a fruit flavored jelly bean gets eaten if and only if it is chosen before all 20 of the licorice flavored jelly beans. Thus $\mathbb{E}(X_1 + \cdots + X_{30}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{30}) = 1/21 + \cdots + 1/21 = (30)(1/21) = 30/21 = 10/7$.
2b. We have $\mathbb{E}(X_iX_j) = P(X_iX_j = 1) = P(X_i = 1)P(X_j = 1 \mid X_i = 1) = (2/22)(1/21)$, since a pair of fruit flavored jelly bean gets eaten if and only if both of the pair are chosen before all 20 of the licorice flavored jelly beans. Also $\mathbb{E}(X_iX_i) = \mathbb{E}(X_i) = 1/21$. Thus $\mathbb{E}((X_1 + \cdots + X_{30})^2) = \mathbb{E}(X_1X_1 + X_1X_2 + \cdots + X_{30}X_{30}) = 30\mathbb{E}(X_1X_1) + 870\mathbb{E}(X_1X_2) = (30)(1/21) + 870(2/22)(1/21) = 400/77$. So $\operatorname{Var}(X_1 + \cdots + X_{30}) = \mathbb{E}((X_1 + \cdots + X_{30})^2) - (\mathbb{E}(X_1 + \cdots + X_{30}))^2 = 400/77 - (10/7)^2 = 1700/539$.

3a. Let X_j indicate if the jth album gets back into its correct cover, i.e., $X_j = 1$ if the jth album gets put back into its correct cover, or $X_j = 0$ otherwise. Thus $\mathbb{E}(X_j) = P(X_j = 1) = 1/10$. So $\mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10}) = 1/10 + \cdots + 1/10 = (10)(1/10) = 1$.
3b. We see $\mathbb{E}(X_iX_j) = P(X_iX_j = 1) = P(X_i = 1 \& X_j = 1) = P(X_i = 1)P(X_j = 1|X_j = 1) = (1/10)(1/9) = 1/90$. Also $\mathbb{E}(X_iX_i) = \mathbb{E}(X_i) = 1/10$. $\mathbb{E}((X_1 + \cdots + X_{10})^2) = \mathbb{E}(X_1X_1 + X_1X_2 + \cdots + X_{10}X_{10}) = 10\mathbb{E}(X_1X_1) + 90\mathbb{E}(X_1X_2) = 10(1/10) + 90(1/90) = 1 + 1 = 2$. So $\operatorname{Var}(X_1 + \cdots + X_{10}) = \mathbb{E}((X_1 + \cdots + X_{10})^2) - (\mathbb{E}(X_1 + \cdots + X_{10}))^2 = 2 - 1^2 = 1$.

4. The random variable X denotes the number of successes in 10 independent trials, each of which has probability $4/24 = 1/6$. So X is a Binomial random variable with parameters $n = 10$ and $p = 1/6$.