STAT/MA 41600
In-Class Problem Set #16: September 23, 2016
Solutions by Mark Daniel Ward

Problem Set 16 Answers

1a. We use X to denote the number of Rhonda's rolls and Y to denote the number of Bernadette's rolls. So X and Y are independent geometric random variables with parameters $1/6$ and $1/4$, respectively. Therefore, we conclude that $\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(-Y) = \text{Var}(X) + \text{Var}(Y) = \frac{5/6}{(1/6)^2} + \frac{3/4}{(1/4)^2} = 42$.
1b. We compute $P(X = Y) = \sum_{n=1}^{\infty} P(X = Y = n) = \sum_{n=1}^{\infty} (5/6)^{n-1}(1/6)(3/4)^{n-1}(1/4) = (1/24)\sum_{n=1}^{\infty}(5/8)^{n-1} = (1/24)(\frac{1}{1-5/8}) = 1/9$.
2. We have

$$\begin{aligned} P(X > Y) &= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}(5/6)^{x-1}(1/6)(3/4)^{y-1}(1/4) \\ &= (1/6)(1/4)\sum_{y=1}^{\infty}(3/4)^{y-1}\sum_{x=y+1}^{\infty}(5/6)^{x-1} \\ &= (1/6)(1/4)\sum_{y=1}^{\infty}(3/4)^{y-1}\frac{(5/6)^y}{1-5/6} \\ &= (5/6)(1/4)\sum_{y=1}^{\infty}(3/4)^{y-1}(5/6)^{y-1} \\ &= (5/6)(1/4)\sum_{y=1}^{\infty}(15/24)^{y-1} \\ &= (5/6)(1/4)\Big(\frac{1}{1-15/24}\Big) \\ &= 5/9 \end{aligned}$$

3a. We have $\mathbb{E}(X-Y) = \mathbb{E}(X) - \mathbb{E}(Y) = \frac{1}{1/4} - \frac{1}{1/5} = -1$.
3b. We have $\text{Var}\,(X-Y) = \text{Var}\,(X) + \text{Var}\,(-Y) = \text{Var}\,(X) + (-1)^2\,\text{Var}\,(Y) = \frac{3/4}{(1/4)^2} + \frac{4/5}{(1/5)^2} = 32$.
4a. Since X is geometric, we have $P(X > 7 \mid X > 5) = P(X > 2) = (3/4)^2 = 9/16$. Alternatively, we could compute $P(X > 7 \mid X > 5) = \frac{P(X>7\ \&\ X>5)}{P(X>5)} = \frac{P(X>7)}{P(X>5)} = \frac{(3/4)^7}{(3/4)^5} = (3/4)^2 = 9/16$.
4b. We compute $P(X > 7 \mid X > Y) = \frac{P(X>7\ \&\ X>Y)}{P(X>Y)}$. Then we have

$$
\begin{aligned}
P(X>Y) &= \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} (3/4)^{x-1}(1/4)(4/5)^{y-1}(1/5) \\
&= (1/4)(1/5) \sum_{y=1}^{\infty} (4/5)^{y-1} \sum_{x=y+1}^{\infty} (3/4)^{x-1} \\
&= (1/5) \sum_{y=1}^{\infty} (4/5)^{y-1}(3/4)^{y} \\
&= (3/4)(1/5) \sum_{y=1}^{\infty} (3/5)^{y-1} = (3/4)(1/5)/(1-3/5) = 3/8
\end{aligned}
$$

Here are two ways to compute the numerator. We could compute:

$$
\begin{aligned}
P(X>7 \;\&\; X>Y) &= \sum_{x=8}^{\infty} \sum_{y=1}^{x-1} (3/4)^{x-1}(1/4)(4/5)^{y-1}(1/5) \\
&= (1/4)(1/5) \sum_{x=8}^{\infty} (3/4)^{x-1} \sum_{y=1}^{x-1} (4/5)^{y-1} \\
&= (1/4) \sum_{x=8}^{\infty} (3/4)^{x-1}(1-(4/5)^{x-1}) \\
&= (1/4) \left[\sum_{x=8}^{\infty} (3/4)^{x-1} - \sum_{x=8}^{\infty} (3/5)^{x-1} \right] \\
&= (3/4)^7 - (5/8)(3/5)^7 = 29692899/256000000 = 0.1160
\end{aligned}
$$

or we could compute

$$
\begin{aligned}
P(X>7 \;\&\; X>Y) &= P(X>Y) - P(X \le 7 \;\&\; X>Y) \\
&= 3/8 - \sum_{x=1}^{7} \sum_{y=1}^{x-1} (3/4)^{x-1}(1/4)(4/5)^{y-1}(1/5) \\
&= 3/8 - (1/4)(1/5) \sum_{x=1}^{7} (3/4)^{x-1} \sum_{y=1}^{x-1} (4/5)^{y-1} \\
&= 3/8 - (1/4) \sum_{x=1}^{7} (3/4)^{x-1}(1-(4/5)^{x-1}) \\
&= 3/8 - (1/4) \left[\sum_{x=1}^{7} (3/4)^{x-1} - \sum_{x=1}^{7} (3/5)^{x-1} \right] \\
&= 3/8 - (1/4)\big[(1-(3/4)^7)/(1-3/4) - (1-(3/5)^7)/(1-3/5)\big] \\
&= 29692899/256000000 = 0.1160
\end{aligned}
$$

Therefore $P(X > 7 \mid X > Y) = (29692899/256000000)/(3/8) = (9897633/32000000) = 0.3093$, i.e., $P(X > 7 \mid X > Y) = (0.1160)/(3/8) = 0.3093$.