STAT/MA 41600
Midterm Exam 1 Answers
Friday, October 9, 2015
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1a. We compute $P(X = Y) = \sum_{n=1}^{\infty} P(X = Y = n) = \sum_{n=1}^{\infty} P(X = n)P(Y = n) = \sum_{n=1}^{\infty}(1/3)(2/3)^{n-1}(2/5)(3/5)^{n-1} = \frac{(1/3)(2/5)}{1-(2/3)(3/5)} = 2/9$.
1b. We compute $P(X > Y) = \sum_{n=1}^{\infty} P(X > Y = n) = \sum_{n=1}^{\infty} P(X > n)P(Y = n) = \sum_{n=1}^{\infty}(2/3)^{n}(2/5)(3/5)^{n-1} = \frac{(2/3)(2/5)}{1-(2/3)(3/5)} = 4/9$.

2. Since X is a Negative Binomial random variable with $r = 6$ and $p = 0.45$ then $\mathbb{E}(X) = r/p = 40/3 = 13.3333$ and $\text{Var}\,(X) = rq/p^2 = 440/27 = 16.2963$. Also $P(X \geq 9) = 1 - P(X < 9) = 1 - P(X = 8) - P(X = 7) - P(X = 6) = 1 - \binom{7}{5}q^2p^6 - \binom{6}{5}qp^6 - \binom{5}{5}p^6 = 371169/390625 = 0.9115$.

3a. Let X_j be a Bernoulli random variable that indicates, for the jth bear that is yellow or blue, whether that particular bear is selected before all of the red bears. So $E[X_j] = P(X_j = 1) = 1/11$. The total number of bears selected before the first red is $X_1 + \cdots + X_{20}$. Thus $\mathbb{E}(X_1 + \cdots + X_{20}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{20}) = 1/11 + \cdots + 1/11 = 20/11 \approx 1.818$.
3b. We compute $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{20})^2) = 20\mathbb{E}(X_1^2) + 380\mathbb{E}(X_1X_2) = (20)(\frac{1}{11}) + (380)(\frac{2}{12})(\frac{1}{11}) = 250/33 \approx 7.576$. So $\text{Var}\,(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 250/33 - (20/11)^2 \approx 4.27$.

4a. The exact expression is $P(X = 4) = \binom{20000}{4}\binom{40000}{6}/\binom{60000}{10} = 0.227625\ldots$ (You did not have to put the decimal value, of course; it is probably way too large for your calculator.)
4b. Since X is approximately Binomial with $n = 10$ and $p = M/N = 20000/60000 = 1/3$, then $P(X = 4)$ is approximately equal to $\binom{10}{4}(1/3)^4(2/3)^6 = 0.227608\ldots$.

5a. Without loss of generality, place one red plate on the table. If all three red plates are to be in a cluster, this first red plate could be the far left, the middle, or the far right of the three in the eventual cluster (i.e., 3 possibilities). Each such possibility has probability $(2/5)(1/4) = 2/20$ of occurring. So the total probability is $2/20+2/20+2/20 = 6/20 = 3/10$. [[Alternatively: Place one red plate. Then there are $\binom{5}{3} = 10$ equally likely ways remaining for the blue plates; in 3 of these ways, the blue plates are adjacent (and therefore the red plates are adjacent too), so the probability is $3/10$.]]
5b. Place one red plate on the table. If all four red plates are to be in a cluster, this first red plate could be the far left, the middle left, the middle right, or the far right of the four in the eventual cluster (i.e., 4 possibilities). Each such possibility has probability $(3/7)(2/6)(1/5) = 1/35$ of occurring. So the total probability is $1/35+1/35+1/35+1/35 = 4/35$. [[Alternatively: Place one red plate. Then there are $\binom{7}{4} = 35$ equally likely ways remaining for the blue plates; in 4 of these ways, the blue plates are adjacent (and therefore the red plates are adjacent too), so the probability is $4/35$.]]

Question 1 was like question #3 on the 9/15/2014 problem set, with some small changes.

Question 2 was like question #5 on the 9/29/2014 problem set, with some small changes.

Question 3 was exactly question #5 on the “more practice problems” for October 5, with the variance included; it was also like question #1 on the 10/5/2015 in-class problem set.

Question 4 was like question #3 on the 10/2/2015 in-class set, with some small changes.

Question 5 was exactly like question #3 on the 8/29/2014 problem set, with 2 and 3 changed to 3 and 4, respectively.