STAT/MA 41600
In-Class Problem Set #40: November 30, 2015
Solutions by Mark Daniel Ward

1a. For $x > 0$, we have $f_X(x) = \int_x^\infty 70e^{-3x-7y}\,dy = 10e^{-10x}$; for $x < 0$, we have $f_X(x) = 0$.
1b. For $x > 0$, we have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{70e^{-3x-7y}}{10e^{-10x}} = 7e^{7x-7y}$ for $y > x$; and $f_{Y|X}(y \mid x) = 0$ for $y \leq x$.
1c. For $x > 0$, we have $\mathbb{E}(Y \mid X = x) = \int_x^\infty (y)(7e^{7x-7y})\,dy = x + 1/7$.
1d. We compute $\mathbb{E}(Y) = \int_0^\infty (x + 1/7)(10e^{-10x})\,dx = 1/10 + 1/7 = 17/70 = 0.2429$.

2a. The conditional mass of X, given $Y = 3$, is $f_{X|Y}(3 \mid 3) = 1/3$; $f_{X|Y}(4 \mid 3) = 2/3$; and $f_{X|Y}(x \mid 3) = 0$ otherwise. So $\mathbb{E}(X \mid Y = 3) = (3)(1/3) + (4)(2/3) = 11/3$.

2b. The conditional mass of Y, given $X = 3$, is $f_{Y|X}(1 \mid 3) = 2/5$; $f_{Y|X}(2 \mid 3) = 2/5$; $f_{Y|X}(3 \mid 3) = 1/5$; and $f_{Y|X}(y \mid 3) = 0$ otherwise. So $\mathbb{E}(Y \mid X = 3) = (1)(2/5) + (2)(2/5) + (3)(1/5) = 9/5$.

3a. For $0 < y < 8$, we have $f_Y(y) = \int_0^{(8-y)/4} 1/8\,dx = (1/8)(8-y)/4 = (8-y)/32$. So we get $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1/8}{(8-y)/32} = 4/(8-y)$. So we conclude $\mathbb{E}(X \mid Y = y) = \int_0^{(8-y)/4} (x)(4/(8-y))\,dx = (8-y)/8$.
3b. For $0 < x < 2$, we have $f_X(x) = \int_0^{8-4x} 1/8\,dy = (2-x)/2$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/8}{(2-x)/2} = 1/(4(2-x))$. So we conclude $\mathbb{E}(Y \mid X = x) = \int_0^{8-4x} (y)1/(4(2-x))\,dy = 4 - 2x$.

4a. Intuitively, if Alice got no reds, then Bob is drawing 2 marbles from a collection of 2 reds and 4 non-reds, so $\mathbb{E}(X \mid Y = 0) = 2/6 + 2/6 = 2/3$.

4b. Intuitively, if Alice got 1 red, then Bob is drawing 2 marbles from a collection of 1 red and 5 non-reds, so $\mathbb{E}(X \mid Y = 1) = 1/6 + 1/6 = 1/3$.

4c. Intuitively, if Alice got 2 reds, then Bob is drawing 2 marbles from a collection of 0 reds and 6 non-reds, so $\mathbb{E}(X \mid Y = 2) = 0/6 + 0/6 = 0$.

4. Here is a more formal way to solve question **4**. We have

$$p_{X,Y}(2,0) = \frac{\binom{6}{0}\binom{2}{2}}{\binom{8}{2}}\frac{\binom{6}{2}\binom{0}{0}}{\binom{6}{2}} = (1/28)(1) = 1/28 \qquad p_{X,Y}(0,2) = \frac{\binom{6}{2}\binom{2}{0}}{\binom{8}{2}}\frac{\binom{4}{0}\binom{2}{2}}{\binom{6}{2}} = (15/28)(1/15) = 1/28$$
$$p_{X,Y}(1,1) = \frac{\binom{6}{1}\binom{2}{1}}{\binom{8}{2}}\frac{\binom{5}{1}\binom{1}{1}}{\binom{6}{2}} = (3/7)(1/3) = 1/7. \qquad p_{X,Y}(1,0) = \frac{\binom{6}{1}\binom{2}{1}}{\binom{8}{2}}\frac{\binom{5}{2}\binom{1}{0}}{\binom{6}{2}} = (3/7)(2/3) = 2/7.$$
$$p_{X,Y}(0,1) = \frac{\binom{6}{2}\binom{2}{0}}{\binom{8}{2}}\frac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}} = (15/28)(8/15) = 2/7. \qquad p_{X,Y}(0,0) = \frac{\binom{6}{2}\binom{2}{0}}{\binom{8}{2}}\frac{\binom{4}{2}\binom{2}{0}}{\binom{6}{2}} = (15/28)(2/5) = 3/14.$$

So $p_Y(0) = 1/28 + 2/7 + 3/14 = 15/28$ and $p_Y(1) = 1/7 + 2/7 = 3/7$ and $p_Y(2) = 1/28$. Thus:

4a. We have $p_{X \mid Y}(x \mid 0) = \frac{p_{X,Y}(x,0)}{p_Y(0)}$. Thus $p_{X \mid Y}(0 \mid 0) = \frac{3/14}{15/28} = 2/5$ and $p_{X \mid Y}(1 \mid 0) = \frac{2/7}{15/28} = 8/15$ and $p_{X \mid Y}(2 \mid 0) = \frac{1/28}{15/28} = 1/15$, so $\mathbb{E}(X \mid Y = 0) = (0)(2/5) + (1)(8/15) + (2)(1/15) = 2/3$.

4b. We have $p_{X \mid Y}(x \mid 1) = \frac{p_{X,Y}(x,1)}{p_Y(1)}$. Thus $p_{X \mid Y}(0 \mid 1) = \frac{2/7}{3/7} = 2/3$ and $p_{X \mid Y}(1 \mid 1) = \frac{1/7}{3/7} = 1/3$, so $\mathbb{E}(X \mid Y = 1) = (0)(2/3) + (1)(1/3) = 1/3$.

4c. We have $p_{X \mid Y}(x \mid 2) = \frac{p_{X,Y}(x,2)}{p_Y(2)}$. Thus $p_{X \mid Y}(0 \mid 2) = \frac{1/28}{1/28} = 1$, so $\mathbb{E}(X \mid Y = 2) = (0)(1) = 0$.