STAT/MA 41600
In-Class Problem Set #39: November 20, 2015
Solutions by Mark Daniel Ward

1. Method #1. We keep track (in order) of the kind of bears that we get. Let X denote the number of red bears selected. For $i = 1, 2, 3, 4, 5$, let $X_i = 1$ if the ith bear selected is red, and $X_i = 0$ otherwise. So $X = X_1 + X_2 + X_3 + X_4 + X_5$. Thus $\text{Var}(X) = \text{Var}(X_1 + \cdots + X_5) = \sum_{i=1}^{5} \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j)$. We have $\text{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = 1/3 - (1/3)^2 = 2/9$. Also $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (3/9)(2/8) - (1/3)(1/3) = -1/36$. So altogether $\text{Var}(X) = (5)(2/9) + (20)(-1/36) = 5/9$.

Method #2. Refer to the red father bear as red bear #1, and the red mother bear as red bear #2, and the red baby bear as red bear #3. For $i = 1, 2, 3$, let $Y_i = 1$ if the ith red bear is selected (at any time, i.e., on any of the five selections), and $Y_i = 0$ otherwise. So we have $X = Y_1 + Y_2 + Y_3$. So $\text{Var}(X) = \text{Var}(Y_1 + Y_2 + Y_3) = \sum_{i=1}^{3} \text{Var}(Y_i) + \sum_{i \neq j} \text{Cov}(Y_i, Y_j)$. We have $\text{Var}(Y_i) = \mathbb{E}(Y_i^2) - (\mathbb{E}(Y_i))^2 = 5/9 - (5/9)^2 = 20/81$. Also $\text{Cov}(Y_i, Y_j) = \mathbb{E}(Y_i Y_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j) = (5/9)(4/8) - (5/9)(5/9) = -5/162$. So altogether $\text{Var}(X) = (3)(20/81) + (6)(-5/162) = 5/9$.

2a. The covariance of X and Y is $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = (2/5)(1/4) - (1/4)(1/4) = 3/80$.

2b. We have $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1/4 - (1/4)^2 = 3/16$. Similarly, $\text{Var}(Y) = 3/16$. So the correlation of X and Y is $\rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{3/80}{\sqrt{(3/16)(3/16)}} = 1/5$.

3a. First we note that $0 \le X \le 2$ and $0 \le Y \le 2$, with the additional constraint that $0 \le X + Y \le 2$. So we have $XY = 1$ if $X = 1$ and $Y = 1$, or otherwise $XY = 0$. (You can try the various combinations of the X and Y, if you do not see this immediately.) Thus $\mathbb{E}(XY) = (1)P(X = 1 \ \&\ Y = 1) + (0)(1 - P(X = 1 \ \&\ Y = 1))$. So $\mathbb{E}(XY) = \frac{\binom{2}{1}\binom{6}{1}}{\binom{8}{2}} \frac{\binom{1}{1}\binom{5}{1}}{\binom{6}{2}} = 1/7$. Thus $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1/7 - (1/2)(1/2) = -3/28$.

3b. We have $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. We calculate $\mathbb{E}(X^2) = (0^2)(6/8)(5/7) + (1^2)((2/8)(6/7) + (6/8)(2/7)) + (2^2)(2/8)(1/7) = 4/7$ and we know $\mathbb{E}(X) = 1/2$, so $\text{Var}(X) = 4/7 - (1/2)^2 = 9/28$. Similarly, $\text{Var}(Y) = 9/28$. So the correlation of X and Y is $\rho(X, Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{-3/28}{\sqrt{(9/28)(9/28)}} = -1/3$.

4a. The density $f_Y(y)$ is $f_Y(y) = \int_0^y 10e^{-3x-2y}\,dx = (10/3)(e^{-2y} - e^{-5y})$ for $y > 0$, and $f_Y(y) = 0$ otherwise.

4b. We have $\mathbb{E}(XY) = \int_0^\infty \int_x^\infty (xy)(10e^{-3x-2y})\,dy\,dx = \int_0^\infty 10xe^{-3x} \int_x^\infty y(e^{-2y})\,dy\,dx = \int_0^\infty 10xe^{-3x}(1/4)(2x+1)e^{-2x}\,dx = \int_0^\infty (5/2)(2x^2 + x)e^{-5x}\,dx = 9/50$.

Also $\mathbb{E}(X) = \int_0^\infty (x)(5e^{-5x})\,dx = 1/5$ and $\mathbb{E}(Y) = \int_0^\infty (y)(10/3)(e^{-2y} - e^{-5y})\,dy = (10/3)((1/2)^2 - (1/5)^2) = 7/10$.

So $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 9/50 - (1/5)(7/10) = 1/25$.