STAT/MA 41600
In-Class Problem Set #32 part 2: November 2, 2015
Solutions by Mark Daniel Ward

1a. We have $P(X+Y<12)=\int_0^{12}\int_0^{12-x}(1/5)e^{-(1/5)x}(1/5)e^{-(1/5)y}\,dy\,dx$. For the inner integral, we focus on the y part of the integrand, and we get: $\int_0^{12-x}(1/5)e^{-(1/5)y}\,dy=1-e^{(-12/5)+(1/5)x}$. So then we have $P(X+Y<12)=\int_0^{12}(1/5)e^{-(1/5)x}(1-e^{(-12/5)+(1/5)x})\,dx=\int_0^{12}(1/5)e^{-(1/5)x}\,dx-\int_0^{12}(1/5)e^{-12/5}\,dx=1-e^{-12/5}-(12/5)e^{-12/5}=1-(17/5)e^{-12/5}=0.6916$.

1b. The calculation is the same, but using "a" instead of 12.

We have $P(X+Y<a)=\int_0^a\int_0^{a-x}(1/5)e^{-(1/5)x}(1/5)e^{-(1/5)y}\,dy\,dx$. For the inner integral, we focus on the y part of the integrand, and we get: $\int_0^{a-x}(1/5)e^{-(1/5)y}\,dy=1-e^{(-1/5)a+(1/5)x}$. So then we have $P(X+Y<a)=\int_0^a(1/5)e^{-(1/5)x}(1-e^{(-1/5)a+(1/5)x})\,dx=\int_0^a(1/5)e^{-(1/5)x}\,dx-\int_0^a(1/5)e^{-(1/5)a}\,dx=1-e^{-(1/5)a}-(a/5)e^{-(1/5)a}=1-(1+\frac{a}{5})e^{-(1/5)a}$.

1c. Yes! If we use $a=12$ in our answer to **1b**, we get $P(X+Y<12)=1-(1+\frac{12}{5})e^{-(1/5)12}=1-(17/5)e^{-12/5}=0.6916$.

2a. We have $0<U<1$, and therefore $-\infty<\ln U<0$, so we conclude $0<-\frac{1}{5}\ln U<\infty$. So X takes on positive real values.

2b. For $a\le 0$, we have $F_X(a)=0$. For $a>0$, we compute $F_X(a)=P(X\le a)=P(-\frac{1}{5}\ln U\le a)=P(\ln U\ge -5a)=P(U\ge e^{-5a})=1-e^{-5a}$.

2c. Using the CDF of X from **2b**, we see that X is an Exponential random variable with $\lambda=5$, i.e., with average $1/5$.

3. We have $P(X-Y>1)=\int_0^\infty\int_{y+1}^\infty e^{-x}e^{-y}\,dx\,dy$. For the inner integral, we focus on the x part of the integrand, and we get: $\int_{y+1}^\infty e^{-x}\,dx=e^{-(y+1)}$. So then we have $P(X-Y>1)=\int_0^\infty e^{-y}e^{-(y+1)}\,dy=\int_0^\infty e^{-2y-1}\,dy=(1/2)e^{-1}$.

Similarly (just switching the roles of x and y), we get $P(Y-X>1)=(1/2)e^{-1}$.

So altogether we have $P(|X-Y|>1)=(1/2)e^{-1}+(1/2)e^{-1}=e^{-1}=0.3679$.

4a. We have $P(X<1)=F_X(1)=1-e^{-(1/2)(1)}=0.3935$.

4b. The probability is $P(X<1,Y<1,Z<1)=P(X<1)P(Y<1)P(Z<1)=(1-e^{-(1/2)(1)})^3=0.0609$.

4c. The probability is $P(X>1,Y>1,Z>1)=P(X>1)P(Y>1)P(Z>1)=(e^{-(1/2)(1)})^3=0.2231$.

4d. The probability is $3(1-e^{-(1/2)(1)})^1(e^{-(1/2)(1)})^2=0.4342$.

4e. The probability is $3(1-e^{-(1/2)(1)})^2(e^{-(1/2)(1)})^1=0.2817$.

4f. The random variable V is a Binomial random variable with parameters $n=3$ and $p=1-e^{-1/2}=0.3935$.