STAT/MA 41600
In-Class Problem Set #29: October 26, 2015
Solutions by Mark Daniel Ward

1a. One method is that we can compute

$$\mathbb{E}(X^2) = \int_0^\infty \int_x^\infty (x^2)(70e^{-3x-7y})\,dy\,dx = \int_0^\infty (x^2)(70e^{-3x}) \int_x^\infty e^{-7y}\,dy\,dx = \int_0^\infty (x^2)(70e^{-3x})(1/7)e^{-7x}\,dx,$$

which simplifies to

$$\mathbb{E}(X^2) = \int_0^\infty (x^2)(10e^{-10x})\,dx = (x^2)(-e^{-10x})\Big|_{x=0}^\infty - \int_0^\infty (-e^{-10x})(2x)\,dx = 0 + 2\int_0^\infty xe^{-10x}\,dx$$

We already computed (in **1** of the last problem set): $10\int_0^\infty xe^{-10x}\,dx = 1/10$, and thus $\mathbb{E}(X^2) = 2\int_0^\infty xe^{-10x}\,dx = (2/10)(1/10) = 2/100$.
1b. We have $\mathrm{Var}\,(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2/100 - (1/10)^2 = 1/100$.

2a. We have $\mathbb{E}(XY) = \int_0^5\int_0^8 (xy)(1/40)\,dy\,dx = \int_0^5 (x)(4/5)\,dx = 10$.
2b. Yes, X and Y are independent. Their joint density $1/40$ can be factored into $1/5$ and $1/8$, and the joint density is defined on a rectangle.
2c. We have $\mathbb{E}(X) = \int_0^5 (x)(1/5)\,dx = 5/2$.
2d. We have $\mathbb{E}(Y) = \int_0^8 (y)(1/8)\,dy = 4$.
Thus, we can use parts **2b**, **2c**, **2d** to double check that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = (5/2)(4) = 10$. (We emphasize that we can only multiply the expected values this way because the X and Y are independent.)
3a. One method is that we can compute

$$\mathbb{E}(X^2) = \int_0^2\int_0^{8-4x} (x^2)(1/8)\,dy\,dx = \int_0^2 (x^2)(1/8)\int_0^{8-4x} 1\,dy\,dx = \int_0^2 (x^2)(1/8)(8-4x)\,dx,$$

which simplifies to

$$\mathbb{E}(X^2) = \int_0^2 (x^2)\Big(\frac{8-4x}{8}\Big)\,dx = 2/3.$$

FYI, if you decided (instead) to just directly use the density of X, namely, $f_X(x) = \frac{8-4x}{8}$ for $0 \le x \le 2$, we get exactly the line above, $\mathbb{E}(X^2) = \int_0^2 (x^2)(\frac{8-4x}{8})\,dx = 2/3$.
3b. One method is that we can compute

$$\mathbb{E}(XY) = \int_0^2\int_0^{8-4x} (xy)(1/8)\,dy\,dx = \int_0^2 (x)(1/8)\int_0^{8-4x} y\,dy\,dx = \int_0^2 (x)(1/8)(8x^2-32x+32)\,dx = 4/3.$$

You could also have changed the order of integration and the bounds, as another possible method of solution.

4a. We have

$$\mathbb{E}(Y^2) = \int_0^\infty (y^2)(5e^{-5y})\,dy = (y^2)(-e^{-5y})\Big|_{y=0}^\infty - \int_0^\infty (2y)(-e^{-5y})\,dy$$

which simplifies to $\mathbb{E}(Y^2) = (2)\int_0^\infty (y)(-e^{-5y})\,dy$. We saw in **4a** from the previous problem set that $5\int_0^\infty (y)(-e^{-5y})\,dy = 1/5$, so it follows that $\mathbb{E}(Y^2) = (2/5)(1/5) = 2/25$.
4b. We have $\mathrm{Var}\,(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/25 - (1/5)^2 = 1/25$.