STAT/MA 41600
In-Class Problem Set #20/#22: October 5, 2015
Solutions by Mark Daniel Ward

Problem Set 20/22 Answers

1a. We have $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{17}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{17})$. Also $\mathbb{E}(X_j) = 1/4$, so it follows that $\mathbb{E}(X) = (17)(1/4) = 17/4 = 4.25$.
1b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{17})^2)$, which has 17 terms of the form $\mathbb{E}(X_j^2)$ and $17^2 - 17 = 272$ terms of the form $\mathbb{E}(X_iX_j)$. Also $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 1/4$ and $\mathbb{E}(X_iX_j) = (2)(1/5)(1/4) = 1/10$. Thus $\mathbb{E}(X^2) = (17)(1/4) + (272)(1/10) = 31.45$. So altogether we have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 31.45 - (4.25)^2 = 13.3875$.

2a. The probability is $\binom{10}{1}\binom{3}{1}\binom{5}{1}\binom{2}{1}/\binom{20}{4} = 20/323 = 0.06192$.
2b. The probability is $(\binom{10}{4} + \binom{5}{4})/\binom{20}{4} = 43/969 = 0.04438$.
2c. The probability is $(\binom{10}{2}\binom{3}{1}\binom{5}{1}+\binom{10}{2}\binom{3}{1}\binom{2}{1}+\binom{10}{2}\binom{5}{1}\binom{2}{1}+\binom{3}{2}\binom{10}{1}\binom{5}{1}+\binom{3}{2}\binom{10}{1}\binom{2}{1}+\binom{3}{2}\binom{5}{1}\binom{2}{1}+\binom{5}{2}\binom{10}{1}\binom{3}{1} + \binom{5}{2}\binom{10}{1}\binom{2}{1} + \binom{5}{2}\binom{3}{1}\binom{2}{1} + \binom{2}{2}\binom{10}{1}\binom{3}{1} + \binom{2}{2}\binom{10}{1}\binom{5}{1} + \binom{2}{2}\binom{3}{1}\binom{5}{1})/\binom{20}{4} = 458/969 = 0.4727$.
2d. The probability is $(\binom{10}{2}\binom{3}{2} + \binom{10}{2}\binom{5}{2} + \binom{10}{2}\binom{2}{2} + \binom{3}{2}\binom{5}{2} + \binom{3}{2}\binom{2}{2} + \binom{5}{2}\binom{2}{2} + \binom{10}{3}\binom{3}{1} + \binom{10}{3}\binom{5}{1} + \binom{10}{3}\binom{2}{1} + \binom{3}{3}\binom{5}{1} + \binom{3}{3}\binom{2}{1} + \binom{5}{3}\binom{2}{1} + \binom{10}{1}\binom{3}{3} + \binom{10}{1}\binom{5}{3} + \binom{3}{1}\binom{5}{3})/\binom{20}{4} = 8/19 = 0.4211$.
[[Indeed, the four probabilities above do sum to exactly 1.]]

3a. We can write $X = X_1 + \cdots + X_{10}$ where $X_j = 1$ if the jth pair has 1 red and 1 green, or $X_j = 0$ otherwise. Then $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{10}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{10})$. Also $\mathbb{E}(X_j) = 10/19$, so it follows that $\mathbb{E}(X) = (10)(10/19) = 100/19 = 5.2632$.
3b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_{10})^2)$, which has 10 terms of the form $\mathbb{E}(X_j^2)$ and $10^2 - 10 = 90$ terms of the form $\mathbb{E}(X_iX_j)$. Also $\mathbb{E}(X_j^2) = \mathbb{E}(X_j) = 10/19$ and $\mathbb{E}(X_iX_j) = (10/19)(9/17) = 90/323$. Thus $\mathbb{E}(X^2) = (10)(10/19) + (90)(90/323) = 9800/323 = 30.3406$. So altogether we have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 30.3406 - (5.2632)^2 = 2.64$.

4a. We have $\mathbb{E}(X) = \mathbb{E}(2Y) = 2\mathbb{E}(Y) = 2(50 + 1)/2 = 51$.
4b. We have $\mathrm{Var}\,(X) = \mathrm{Var}\,(2Y) = 4\,\mathrm{Var}\,(Y) = 4(50^2 - 1)/12 = 833$.