STAT/MA 41600
In-Class Problem Set #18: September 30, 2015
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Problem Set 18 Answers

1a. Since X and Y are independent Poisson random variables, then Z is a Poisson random variable too. We have $\mathbb{E}(Z) = \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 2+2 = 4$. We have $P(Z \leq 3) = p_Z(0)+p_Z(1)+p_Z(2)+p_Z(3) = e^{-4}+4e^{-4}+8e^{-4}+\frac{32}{3}e^{-4} = 0.0183+0.0733+0.1465+0.1954 = 0.4335$.
1b. We calculate a few values of the probability mass function of X, and we find that $p_X(x)$ attains its maximum when $x = 5$; indeed, we have $p_X(5) = e^{-5.3}(5.3)^5/5! = 0.1740$.

2a. We have

$$\begin{aligned} P(X > 4 \mid X > 2) &= \frac{P(X > 4 \ \&\ X > 2)}{P(X > 2)} = \frac{P(X > 4)}{P(X > 2)} = \frac{1 - P(X \leq 4)}{1 - P(X \leq 2)} \\ &= \frac{1 - 0.1108 - 0.2438 - 0.2681 - 0.1966 - 0.1082}{1 - 0.1108 - 0.2438 - 0.2681} = 0.1922 \end{aligned}$$

2b. We have

$$P(X \leq 1 \mid X \leq 3) = \frac{P(X \leq 1 \ \&\ X \leq 3)}{P(X \leq 3)} = \frac{P(X \leq 1)}{P(X \leq 3)} = \frac{0.1108 + 0.2438}{0.1108 + 0.2438 + 0.2681 + 0.1966} = 0.433$$

3a. The exact expression is $\binom{5000000}{4}\left(\frac{1}{2000000}\right)^4\left(\frac{1999999}{2000000}\right)^{5000000-4} = 0.133601909\ldots$ (You do not need this last number in your answer; it takes a computer to approximate the answer.)
3b. The actual number of winners is a Binomial random variable with $n = 5000000$ and $p = 1/2000000$. So n is large and $np(1-p)$ is roughly $5/2$ which is a moderate size number, i.e., not too far from 1. So the number of winners is approximately Poisson with $\lambda = np = 5/2$. So the probability of 4 winners is approximately $e^{-5/2}(5/2)^4/4! = 0.133601886\ldots$.

4. We have

$$\begin{aligned} E((X)(X-1)(X-2)) &= \sum_{x=0}^{\infty}(x)(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=3}^{\infty}(x)(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{because } x = 0, 1, 2 \text{ terms are themselves } 0 \\ &= \sum_{x=3}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-3)!} \qquad \text{divide out by } x \text{ and } x-1 \text{ and } x-2 \\ &= \lambda^3 e^{-\lambda}\sum_{x=3}^{\infty}\frac{\lambda^{x-3}}{(x-3)!} \qquad \text{factor out } e^{-\lambda} \text{ and } \lambda^3 \\ &= \lambda^3 e^{-\lambda}\left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \ldots\right) \\ &= \lambda^3 e^{-\lambda} e^{\lambda} \\ &= \lambda^3 \end{aligned}$$