STAT/MA 41600
In-Class Problem Set #15: September 23, 2015
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Problem Set 15 Answers

1a. We compute that $P(X = Y) = P(X = Y = 0) + P(X = Y = 1) + P(X = Y = 2) + P(X = Y = 3) = \binom{3}{0}(9/10)^0(1/10)^3\binom{3}{0}(9/10)^0(1/10)^3 + \binom{3}{1}(9/10)^1(1/10)^2\binom{3}{1}(9/10)^1(1/10)^2 + \binom{3}{2}(9/10)^2(1/10)^1\binom{3}{2}(9/10)^2(1/10)^1 + \binom{3}{3}(9/10)^3(1/10)^0\binom{3}{3}(9/10)^3(1/10)^0$ which simplifies to $P(X = Y) = 1/1000000 + 729/1000000 + 59049/1000000 + 531441/1000000 = 29561/50000 = 0.59122$.

1b. We have $P(X \neq Y) = 1 - P(X = Y) = 1 - 29561/50000 = 20439/50000$, or written with decimals, this is $P(X \neq Y) = 1 - 0.59122 = 0.40878$. By the symmetry of X and Y, we have $P(X > Y) = P(Y > X)$ and thus $P(X > Y) = P(X \neq Y)/2 = (20439/50000)(1/2) = 20439/100000 = 0.20439$.

1c. Similarly to 1b, we have $P(Y > X) = 20439/100000 = 0.20439$.

2a. Let $X_i = 1$ if the ith card is isolated, and $X_i = 0$ otherwise. Then $\mathbb{E}(X_i) = P(X_i = 1) = (10/14)(9/13) = 45/91 = 0.4945$. So $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{15}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{15}) = (15)(45/91) = 675/91 = 7.4176$.

2b. Let $Y_i = 1$ if the ith card is semi-happy, and $Y_i = 0$ otherwise. Then $\mathbb{E}(Y_i) = P(Y_i = 1) = (10/14)(4/13) + (4/14)(10/13) = 40/91 = 0.43956$. So $\mathbb{E}(Y) = \mathbb{E}(Y_1 + \cdots + Y_{15}) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_{15}) = (15)(40/91) = 600/91 = 6.5934$.

2c. Let $Z_i = 1$ if the ith card is joyous, and $Z_i = 0$ otherwise. Then $\mathbb{E}(Z_i) = P(Z_i = 1) = (4/14)(3/13) = 6/91 = 0.06593$. So $\mathbb{E}(Z) = \mathbb{E}(Z_1 + \cdots + Z_{15}) = \mathbb{E}(Z_1) + \cdots + \mathbb{E}(Z_{15}) = (15)(6/91) = 90/91 = 0.9890$.

[We verify, by the way, that $\mathbb{E}(X) + \mathbb{E}(Y) + \mathbb{E}(Z) = 7.4176 + 6.5934 + 0.9890 = 15$.]

3a. Yes, $W = X + Y$ is a Binomial random variable. To see this, notice that $X = X_1 + \cdots + X_5$ and $Y = Y_1 + \cdots + Y_5$ where the 10 Bernoulli random variables $X_1, \ldots, X_5, Y_1, \ldots, Y_5$ are independent and each have $p = 0.35$. So W is the sum of 10 independent Bernoulli random variables with $p = 0.35$, so W is a Binomial random variable with $n = 10$ and $p = 0.35$.

3b. No, U is not a Binomial random variable. An easy way to see this is to note, for instance, that if $X = 0$ and $Y = 3$, then $U = -3$, so U can take on negative values. Binomial random variables only take on values $0, \ldots, n$ for some n, and so U cannot be a Binomial random variable.

4a. Since X is a Binomial random variable with $n = 5$ and $p = 2/6 = 1/3$, then $\mathbb{E}(X) = np = 5/3$. Since Y is a Binomial random variable with $n = 5$ and $p = 1/2$, then $\mathbb{E}(Y) = np = 5/2$. Thus $\mathbb{E}(X - Y) = \mathbb{E}(X) - \mathbb{E}(Y) = 5/3 - 5/2 = -5/6$.

4b. For this part (but not for the last part), we need to use the fact that X and Y are independent. So we have $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + (-1)^2\,\mathrm{Var}(Y) = (5)(1/3)(2/3) + (5)(1/2)(1/2) = 85/36 = 2.3611$.