STAT/MA 41600
Midterm Exam 1 Answers
Friday, October 5, 2018
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1a. We let $X_j = 1$ if the jth child chosen is a girl, and $X_j = 0$ otherwise. Therefore, we have $\mathbb{E}(X_j) = (3/6)(1) + (3/6)(0) = 1/2$. We conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 1/2 + 1/2 + 1/2 = 3/2$.
1b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2 + X_3)^2) = 3\mathbb{E}(X_1^2) + 6\mathbb{E}(X_1X_2)$. We have $\mathbb{E}(X_1^2) = \mathbb{E}(X_1) = 3/6 = 1/2$ and $\mathbb{E}(X_1X_2) = P(X_1X_2 = 1) = P(X_1 = 1)P(X_2 = 1 \mid X_1 = 1) = (3/6)(2/5) = 1/5$. So we conclude that $\mathbb{E}(X^2) = (3)(1/2) + (6)(1/5) = 27/10$.
We compute $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 27/10 - (3/2)^2 = 9/20$.

2. Each X_i is a Geometric random variable. We see that X_1 has $p = 1$ so $\mathbb{E}(X_1) = 1$, and X_2 has $p = 5/6$ so $\mathbb{E}(X_2) = 6/5$, and X_3 has $p = 4/6$ so $\mathbb{E}(X_3) = 6/4$, etc., etc. Altogether, we have $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_6) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_6) = \frac{1}{6/6} + \frac{1}{5/6} + \frac{1}{4/6} + \frac{1}{3/6} + \frac{1}{2/6} + \frac{1}{1/6} = 147/10 = 14.7$.

3a. The number of defects is a Binomial random variable with $n = 3{,}000{,}000$ and $p = 1/1{,}000{,}000$, so the exact expression for the probability that there are 4 or fewer defects is

$$\sum_{x=0}^{4} \binom{3{,}000{,}000}{x} \left(\frac{1}{1{,}000{,}000}\right)^x \left(\frac{999{,}999}{1{,}000{,}000}\right)^{3{,}000{,}000-x}.$$

3b. The distribution of the number of defects is approximately Poisson with $\lambda = np = 3$. So the approximation to the probability above is $\sum_{x=0}^{4} e^{-3}3^x/x! = e^{-3}(1 + 3 + 9/2 + 27/6 + 81/24) = e^{-3}131/8 = 0.8153$.

4. We can think about a sequence of independent successes (with probability 3/5) and/or failures (with probability 2/5). In part **4a**, we just need to have 10 failures in a row. In part **4b**, we need the third success to occur after the first 10 tries, so we could have 10 failures in a row, or 9 failures and 1 success (in any order), or 8 failures and 2 successes (in any order). (Alternatively, we could use the probability mass function of the Negative Binomial for **4b**.)
4a. We have $P(X > 10) = (2/5)^{10}$.
4b. We have $P(X + Y + Z > 10) = (2/5)^{10} + \binom{10}{1}(2/5)^9(3/5) + \binom{10}{2}(2/5)^8(3/5)^2$.

5. Let A be the event that Alice gets no heads. Let B_n be the event that Bob rolls n times. Then we compute

$$P(A) = \sum_{n=1}^{\infty} P(A \cap B_n) = \sum_{n=1}^{\infty} P(A \mid B_n)P(B_n) = \sum_{n=1}^{\infty} (1/2)^n (5/6)^{n-1}(1/6) = (1/2)(1/6)\sum_{n=1}^{\infty}(5/12)^{n-1}.$$

So we conclude that $P(A) = (1/2)(1/6)/(1 - 5/12) = 1/7$.

The problems come from:
(1) 2017, PS 12; (2) 2015, PS 17; (3) 2018, PS 18; (4) 2015, PS 16; (5) 2018, PS 5