STAT/MA 41600
In-Class Problem Set #39 part 2: November 16, 2018
Solutions by Mark Daniel Ward

Problem Set 39 part 2 Answers

1. We have $\mathbb{E}(XY) = \int_0^\infty \int_x^\infty (xy)(1/750)e^{-(x/150+y/30)}\,dy\,dx$.

For the inner integral (only), integration by parts with $u = y$ and $dv = (x)(1/750)e^{-(x/150+y/30)}$, and thus $du = dy$ and $v = -(x)(1/25)e^{-(x/150+y/30)}$ gives us $\int_x^\infty (xy)(1/750)e^{-(x/150+y/30)}\,dy = -(xy)(1/25)e^{-(x/150+y/30)}|_{y=x}^{\infty} - \int_x^\infty -(x)(1/25)e^{-(x/150+y/30)}\,dy = (x^2)(1/25)e^{-x/25} + (x)(6/5)e^{-x/25}$.

So altogether we have $\mathbb{E}(XY) = \int_0^\infty \left((x^2)(1/25)e^{-x/25} + (x)(6/5)e^{-x/25}\right) dx$. The first term is equal to $(2)(25^2)$, since it is equal to $\mathbb{E}(X^2)$, where X is an exponential random variable with $\lambda = 1/25$. The second term can be rewritten as $\int_0^\infty (x)(6/5)e^{-x/25}\,dx = 30\int_0^\infty (x)(1/25)e^{-x/25}\,dx$, which equals $(30)(25)$, by again using our knowledge of the mean of an exponential random variable with $\lambda = 1/25$. So altogether we have $\mathbb{E}(XY) = (2)(25^2) + (30)(25) = 2000$.

2a. For the inner integral (only), we get
$\int_x^\infty (x)(1/750)e^{-(x/150+y/30)}\,dy = -(x)(1/25)e^{-(x/150+y/30)}|_{y=x}^{\infty} = (x)(1/25)e^{-x/25}$.

Now we have $\mathbb{E}(X) = \int_0^\infty (x^2)(1/25)e^{-x/25}\,dx = 25$, using our knowledge about the mean of exponential random variables.

2b. We have $\mathbb{E}(Y) = \int_0^\infty \int_x^\infty (y)(1/750)e^{-(x/150+y/30)}\,dy\,dx$.

For the inner integral (only), integration by parts with $u = y$ and $dv = (1/750)e^{-(x/150+y/30)}$, and thus $du = dy$ and $v = -(1/25)e^{-(x/150+y/30)}$ gives us $\int_x^\infty (y)(1/750)e^{-(x/150+y/30)}\,dy = -(y)(1/25)e^{-(x/150+y/30)}|_{y=x}^{\infty} - \int_x^\infty -(1/25)e^{-(x/150+y/30)}\,dy = (x)(1/25)e^{-x/25} + (6/5)e^{-x/25}$.

So we get $\mathbb{E}(Y) = \int_0^\infty \left((x)(1/25)e^{-x/25} + (6/5)e^{-x/25}\right) dx$. The first term is 25 (by using our knowledge about the mean of exponential random variables). The second term is $\int_0^\infty (6/5)e^{-x/25}\,dx = 30\int_0^\infty (1/25)e^{-x/25}\,dx = 30$. So altogether we have $\mathbb{E}(Y) = 25 + 30 = 55$.

3. As before, we compute $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = (12/52)(11/51) - (3/13)^2 = -10/2873$. We also have $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2 = \mathbb{E}(X_1) - (\mathbb{E}(X_1))^2 = 3/13 - (3/13)^2 = 30/169$. So altogether we have $\rho(X_1, X_2) = \mathrm{Cov}(X_1, X_2)/\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)} = (-10/2873)/\sqrt{(30/169)(30/169)} = -1/51$.

4. As before, we compute $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = 12/90 - (1/3)^2 = 1/45$. We also have $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2 = \mathbb{E}(X_1) - (\mathbb{E}(X_1))^2 = 1/3 - (1/3)^2 = 2/9$. So altogether we have $\rho(X_1, X_2) = \mathrm{Cov}(X_1, X_2)/\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)} = (1/45)/\sqrt{(2/9)(2/9)} = 1/10$.