STAT/MA 41600
In-Class Problem Set #39: November 12, 2018
Solutions by Mark Daniel Ward

Problem Set 39 Answers

1a. We compute $\mathrm{Cov}(X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 3/13 - (3/13)(3/13) = 30/169$.
1b. We compute $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_i) = (3/13)(3/13) - (3/13)(3/13) = 0$. It is also possible to observe $\mathrm{Cov}(X_i, X_j) = 0$ since X_i and X_j are independent in this setup (since we pick cards with replacement).
1c. The variance is $\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathrm{Cov}(X_1 + \cdots + X_5, X_1 + \cdots + X_5) = 5\mathrm{Cov}(X_1, X_1) + 20\mathrm{Cov}(X_1, X_2) = 5(30/169) + 20(0) = 150/169$. Yes, this agrees with 1cd on Problem Set #12.

2a. We compute $\mathrm{Cov}(X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 3/13 - (3/13)(3/13) = 30/169$.
2b. We compute $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_i) = (12/52)(11/51) - (3/13)^2 = -10/2873$.
2c. The variance is $\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathrm{Cov}(X_1 + \cdots + X_5, X_1 + \cdots + X_5) = 5\mathrm{Cov}(X_1, X_1) + 20\mathrm{Cov}(X_1, X_2) = 5(30/169) + 20(-10/2873) = 2350/2873$. Yes, this agrees with 2c on P.S. #12.

3a. We compute
$\mathrm{Cov}(X_1, X_1) = \mathbb{E}(X_1 X_1) - \mathbb{E}(X_1)\mathbb{E}(X_1) = 1 - (1)(1) = 0$.
$\mathrm{Cov}(X_2, X_2) = \mathbb{E}(X_2 X_2) - \mathbb{E}(X_2)\mathbb{E}(X_4) = (3/4)^3 - (3/4)^3(3/4)^3 = 999/4096$.
$\mathrm{Cov}(X_3, X_3) = \mathbb{E}(X_3 X_3) - \mathbb{E}(X_3)\mathbb{E}(X_4) = (2/4)^3 - (2/4)^3(2/4)^3 = 7/64$.
$\mathrm{Cov}(X_4, X_4) = \mathbb{E}(X_4 X_4) - \mathbb{E}(X_4)\mathbb{E}(X_4) = (1/4)^3 - (1/4)^3(1/4)^3 = 63/4096$.
3b. We compute
$\mathrm{Cov}(X_1, X_j) = \mathbb{E}(X_1 X_j) - \mathbb{E}(X_1)\mathbb{E}(X_i) = 0$ for any j (just notice that X_1 is always 1, so X_1 is independent from the other X_j's)
$\mathrm{Cov}(X_2, X_3) = \mathbb{E}(X_2 X_3) - \mathbb{E}(X_2)\mathbb{E}(X_3) = \mathbb{E}(X_3) - \mathbb{E}(X_2)\mathbb{E}(X_3) = (2/4)^3 - (3/4)^3(2/4)^3 = 37/512$
$\mathrm{Cov}(X_2, X_4) = \mathbb{E}(X_2 X_4) - \mathbb{E}(X_2)\mathbb{E}(X_4) = \mathbb{E}(X_4) - \mathbb{E}(X_2)\mathbb{E}(X_4) = (1/4)^3 - (3/4)^3(1/4)^3 = 37/4096$
$\mathrm{Cov}(X_3, X_4) = \mathbb{E}(X_3 X_4) - \mathbb{E}(X_3)\mathbb{E}(X_4) = \mathbb{E}(X_4) - \mathbb{E}(X_3)\mathbb{E}(X_4) = (1/4)^3 - (2/4)^3(1/4)^3 = 7/512$
3c. The variance is

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Cov}(X, X) = \mathrm{Cov}(X_1 + \cdots + X_4, X_1 + \cdots + X_4) \\
&= \mathrm{Cov}(X_1, X_1) + \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_1, X_4) \\
&\quad + \mathrm{Cov}(X_2, X_1) + \mathrm{Cov}(X_2, X_2) + \mathrm{Cov}(X_2, X_3) + \mathrm{Cov}(X_2, X_4) \\
&\quad + \mathrm{Cov}(X_3, X_1) + \mathrm{Cov}(X_3, X_2) + \mathrm{Cov}(X_3, X_3) + \mathrm{Cov}(X_3, X_4) \\
&\quad + \mathrm{Cov}(X_4, X_1) + \mathrm{Cov}(X_4, X_2) + \mathrm{Cov}(X_4, X_3) + \mathrm{Cov}(X_4, X_4) \\
&= 0 + 0 + 0 + 0 \\
&\quad + 0 + 999/4096 + 37/512 + 37/4096 \\
&\quad + 0 + 37/512 + 7/64 + 7/512 \\
&\quad + 0 + 37/4096 + 7/512 + 63/4096 \\
&= 143/256
\end{aligned}$$

Yes, this agrees with 3c on P.S. #12.

4a. We compute $\mathrm{Cov}(X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 1/3 - (1/3)(1/3) = 2/9$.
4b. We compute $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 12/90 - (1/3)^2 = 1/45$.
4c. The variance is $\mathrm{Var}(X) = \mathrm{Cov}(X, X) = \mathrm{Cov}(X_1 + \cdots + X_3, X_1 + \cdots + X_3) = 3\mathrm{Cov}(X_1, X_1) + 6\mathrm{Cov}(X_1, X_2) = 3(2/9) + 6(1/45) = 4/5$. Yes, this agrees with 4c on P.S. #12.