STAT/MA 41600
In-Class Problem Set #27: October 17, 2018
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Problem Set 27 Answers

1. We know that $P(Y > 90 \mid X = 35) = \int_{90}^{\infty} f_{Y|X}(y \mid 35)\,dy = \int_{90}^{\infty} \frac{f_{X,Y}(35,y)}{f_X(35)}\,dy$.

For the denominator, we note that, for $x > 0$, the density of X is

$$f_X(x) = \int_x^{\infty} \frac{1}{750} e^{-(x/150+y/30)}\,dy = -\frac{1}{25} e^{-(x/150+y/30)}\Big|_{y=x}^{\infty} = \frac{1}{25} e^{-(x/150+x/30)} = \frac{1}{25} e^{-x/25}.$$

Therefore, we get $f_X(35) = \frac{1}{25} e^{-35/25}$. So we have

$$P(Y > 90 \mid X = 35) = \int_{90}^{\infty} f_{Y|X}(y \mid 35)\,dy = \int_{90}^{\infty} \frac{\frac{1}{750} e^{-(35/150+y/30)}}{\frac{1}{25} e^{-35/25}}\,dy = \int_{90}^{\infty} \frac{1}{30} e^{(7/6-y/30)}\,dy,$$

and we conclude that $P(Y > 90 \mid X = 35) = -e^{(7/6-y/30)}\big|_{y=90}^{\infty} = e^{(7/6-90/30)} = e^{-11/6} = 0.1599$.

2. We have $P(Y > 90 \mid X > 35) = \frac{P(Y>90 \ \&\ X>35)}{P(X>35)} = \frac{\int_{90}^{\infty}\int_{35}^{y} \frac{1}{750} e^{-(x/150+y/30)}\,dx\,dy}{\int_{35}^{\infty}\int_{x}^{\infty} \frac{1}{750} e^{-(x/150+y/30)}\,dy\,dx}$. We compute

$$\int_{90}^{\infty}\int_{35}^{y} \frac{1}{750} e^{-(x/150+y/30)}\,dx\,dy = \int_{90}^{\infty} -\frac{1}{5} e^{-(x/150+y/30)}\Big|_{x=35}^{y}\,dy = \int_{90}^{\infty} \frac{1}{5}\Big(e^{-(y+7)/30} - e^{-y/25}\Big)\,dy$$

so the numerator is $\frac{1}{5}\big(-30e^{-(y+7)/30} + 25e^{-y/25}\big)\big|_{y=90}^{\infty} = 6e^{-97/30} - 5e^{-18/5}$. We also have

$$\int_{35}^{\infty}\int_{x}^{\infty} \frac{1}{750} e^{-(x/150+y/30)}\,dy\,dx = \int_{35}^{\infty} -\frac{1}{25} e^{-(x/150+y/30)}\Big|_{y=x}^{\infty}\,dx = \int_{35}^{\infty} \frac{1}{25} e^{-x/25}\,dx = e^{-7/5}.$$

So we conclude that $P(Y > 90 \mid X > 35) = 6e^{-11/6} - 5e^{-11/5} = 0.4053$.

3a. We have $P(X > 3 \mid Y = 1) = \int_3^{25/3} f_{X|Y}(x \mid 1)\,dx = \int_3^{25/3} \frac{f_{X,Y}(x,1)}{f_Y(1)}\,dx$. We note that $f_Y(1) = \int_0^{25/3} 1/30\,dx = 5/18$. So we have $P(X > 3 \mid Y = 1) = \int_3^{25/3} \frac{1/30}{5/18}\,dx = \int_3^{25/3} \frac{3}{25}\,dx = (3/25)(25/3 - 3) = 16/25 = 0.64$.

3b. We have $P(X > 3 \mid Y > 1) = \frac{P(X>3\ \&\ Y>1)}{P(Y>1)} = \frac{\int_3^{25/3}\int_1^{-3x/5+6} 1/30\,dy\,dx}{\int_0^{25/3}\int_1^{-3x/5+6} 1/30\,dy\,dx} = \frac{\int_3^{25/3}(1/30)(-\frac{3}{5}x+5)\,dx}{\int_0^{25/3}(1/30)(-\frac{3}{5}x+5)\,dx} = \frac{(1/30)(-\frac{3}{10}x^2+5x)|_{x=3}^{25/3}}{(1/30)(-\frac{3}{10}x^2+5x)|_{x=0}^{25/3}} = \frac{(-\frac{3}{10}(25/3)^2+5(25/3)+\frac{3}{10}(3)^2-5(3))}{(-\frac{3}{10}(25/3)^2+5(25/3))} = 256/625 = 0.4096$.

An alternative way to see this is to notice that the probability density function is constant on the given triangle (otherwise this observation will not work), and the area of the triangle with $X > 3$ and $Y > 1$ is $(1/2)(25/3-3)(21/5)$ and the area of the triangle with $Y > 1$ is $(1/2)(25/3)(5)$, so the desired probability is $\frac{(1/2)(25/3-3)(21/5-1)}{(1/2)(25/3)(5)} = 256/625 = 0.4096$.

4. We compute $P(X + Y \le 2) = \int_0^2 \int_0^{2-x} (1/36)(3-x)(4-y)\,dy\,dx = \int_0^2 (1/36)(3-x)(4y - y^2/2)|_{y=0}^{2-x}\,dx = \int_0^2 (1/36)(3-x)(6-2x-x^2/2)\,dx = \int_0^2 (1/36)(18-12x+x^2/2+x^3/2)\,dx = (1/36)(18x - 6x^2 + x^3/6 + x^4/8)|_{x=0}^2 = (1/36)(36 - 24 + 8/6 + 2) = 23/54$.