STAT/MA 41600
In-Class Problem Set #26: October 15, 2018
Solutions by Mark Daniel Ward

Problem Set 26 Answers

1a. The random variables X and Y are dependent because, when X is positive, we have $Y > X$.
1b. We calculate $P(Y > 2X) = \int_0^\infty \int_{2x}^\infty \frac{1}{750} e^{-(x/150+y/30)}\,dy\,dx = \frac{1}{750}\int_0^\infty e^{-x/150}\int_{2x}^\infty e^{-y/30}\,dy\,dx = \frac{1}{750}\int_0^\infty e^{-x/150}(30)e^{-2x/30}\,dx = \frac{1}{25}\int_0^\infty e^{-11x/150}\,dx = \frac{6}{11}$.

2a. We have $1 = \int_0^\infty \int_0^\infty k e^{-(x/150+y/30)}\,dy\,dx = k\int_0^\infty e^{-y/30}\,dy \int_0^\infty e^{-x/150}\,dx = 4500k$, so $k = \frac{1}{4500}$.
2b. Yes, X and Y are independent because the joint probability density function $f_{X,Y}(x,y) = \frac{1}{4500}e^{-y/30}e^{-x/150}$ can be factored as some x stuff times some y stuff.
2c. The density of X is $f_X(x) = \int_0^\infty \frac{1}{4500}e^{-y/30}e^{-x/150}\,dy = -\frac{1}{150}e^{-y/30}e^{-x/150}|_{y=0}^{\infty} = \frac{1}{150}e^{-x/150}$ for $x > 0$, and $f_X(x) = 0$ otherwise.
2d. The density of Y is $f_Y(y) = \int_0^\infty \frac{1}{4500}e^{-y/30}e^{-x/150}\,dx = -\frac{1}{30}e^{-y/30}e^{-x/150}|_{x=0}^{\infty} = \frac{1}{30}e^{-y/30}$ for $y > 0$, and $f_Y(y) = 0$ otherwise.

3a. The random variables X and Y are dependent, because their density is not defined on rectangles. As another way to see this, e.g., we note that if $X > 5$, then we necessarily have $Y < 3$.
3b. We compute $P(5Y < 3X) = P(Y < \frac{3}{5}X) = \int_0^3 \int_{5y/3}^{10-5y/3} 1/30\,dx\,dy = \int_0^3 x/30|_{x=5y/3}^{10-5y/3}\,dy = \int_0^3 (10 - 5y/3 - 5y/3)/30\,dy = \int_0^3 (1 - y/3)/3\,dy = (y - y^2/6)/3|_{y=0}^3 = (3 - 3^2/6)/3 = 1/2$.

An alternative way to see this is to notice that the probability density function is constant on the given triangle (otherwise this observation will not work), and the area of the entire triangle is 30, and the area of the desired triangle is 15, so the desired probability is $15/30 = 1/2$.

4a. We have $1 = \int_0^3 \int_0^4 k(3-x)(4-y)\,dy\,dx = k(3x - x^2/2)|_{x=0}^3 (4y - y^2/2)|_{y=0}^4 = k(9 - 9/2)(16 - 16/2) = k(9/2)(16/2) = 36k$, so $k = \frac{1}{36}$.
4b. Yes, X and Y are independent because the joint probability density function $f_{X,Y}(x,y) = \frac{1}{36}(3-x)(4-y)$ can be factored as some x stuff times some y stuff.
4c. The density of X is $f_X(x) = \int_0^4 \frac{1}{36}(3-x)(4-y)\,dy = \frac{1}{36}(3-x)(4y - y^2/2)|_{y=0}^4 = \frac{1}{36}(3-x)(16 - 16/2) = \frac{2}{9}(3-x)$ for $0 < x < 3$, and $f_X(x) = 0$ otherwise.
4d. The density of Y is $f_Y(y) = \int_0^3 \frac{1}{36}(3-x)(4-y)\,dx = \frac{1}{36}(3x - x^2/2)(4-y)|_{x=0}^3 = \frac{1}{36}(9 - 9/2)(4 - y) = \frac{1}{8}(4-y)$ for $0 < y < 4$, and $f_Y(y) = 0$ otherwise.