STAT/MA 41600

In-Class Problem Set #12: September 17, 2018

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Problem Set 12 Answers

1a. We compute $\mathbb{E}(X^2) = (0^2)(40/52)^5 + (1^2)(5)(40/52)^4(12/52) + (2^2)(10)(40/52)^3(12/52)^2 + (3^2)(10)(40/52)^2(12/52)^3 + (4^2)(5)(40/52)(12/52)^4 + (5^2)(12/52)^5 = 375/169$.

1b. The twenty five terms are $X^2 = (X_1 + \cdots + X_5)^2 = X_1X_1 + X_1X_2 + \cdots + X_5X_5$, so we get $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_5)^2) = \mathbb{E}(X_1X_1) + \mathbb{E}(X_1X_2) + \cdots + \mathbb{E}(X_5X_5)$. By symmetry, this simplifies to $\mathbb{E}(X^2) = 5\mathbb{E}(X_1X_1) + 20\mathbb{E}(X_1X_2)$. We have $X_1X_1 = X_1$ (since X_1 is an indicator random variable), so we get $\mathbb{E}(X_1X_1) = \mathbb{E}(X_1) = 12/52$. We also have $\mathbb{E}(X_1X_2) = (12/52)^2$. So altogether we get $\mathbb{E}(X^2) = (5)(12/52) + (20)(12/52)^2 = 375/169$.

1c. We get $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 375/169 - (15/13)^2 = 150/169$.

1d. Since the X_j's are independent, we can add the variances, i.e., $\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_5) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_5)$, which equals $5\,\mathrm{Var}(X_1)$ by symmetry. We have $\mathrm{Var}(X_1) = \mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2 = \mathbb{E}(X_1) - (\mathbb{E}(X_1))^2 = 12/52 - (12/52)^2 = 30/169$. So we conclude that $\mathrm{Var}(X) = (5)(30/169) = 150/169$.

2a. We have $\mathbb{E}(X^2) = (0^2)(2109/8330) + (1^2)(703/1666) + (2^2)(209/833) + (3^2)(55/833) + (4^2)(165/21658) + (5^2)(33/108290) = 475/221$.

2b. The twenty five terms are $X^2 = (X_1 + \cdots + X_5)^2 = X_1X_1 + X_1X_2 + \cdots + X_5X_5$, so we get $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_5)^2) = \mathbb{E}(X_1X_1) + \mathbb{E}(X_1X_2) + \cdots + \mathbb{E}(X_5X_5)$. By symmetry, this simplifies to $\mathbb{E}(X^2) = 5\mathbb{E}(X_1X_1) + 20\mathbb{E}(X_1X_2)$. We have $X_1X_1 = X_1$ (since X_1 is an indicator random variable), so we get $\mathbb{E}(X_1X_1) = \mathbb{E}(X_1) = 12/52$. We also have $\mathbb{E}(X_1X_2) = (12/52)(11/51)$. So altogether we get $\mathbb{E}(X^2) = (5)(12/52) + (20)(12/52)(11/51) = 475/221$.

2c. We get $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 475/221 - (15/13)^2 = 2350/2873$.

3a. We have $\mathbb{E}(X^2) = (1^2)(37/64) + (2^2)(19/64) + (3^2)(7/64) + (4^2)(1/64) = 3$.

3b. We have $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_4)^2)$, which has:
6 terms of the form $\mathbb{E}(X_iX_4) = \mathbb{E}(X_4)$ for $i < 4$; 4 terms of the form $\mathbb{E}(X_iX_3) = \mathbb{E}(X_3)$ for $i < 3$; 2 terms of the form $\mathbb{E}(X_iX_2) = \mathbb{E}(X_2)$ for $i < 2$; and of course the terms of the form $\mathbb{E}(X_j^2) = \mathbb{E}(X_j)$.

So we get $\mathbb{E}(X^2) = (7)(1/4)^3 + (5)(2/4)^3 + (3)(3/4)^3 + (1)(1) = 3$.

3c. We get $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 3 - (25/16)^2 = 143/256$.

4a. We have $\mathbb{E}(X^2) = (0^2)(1/3) + (1^2)(2/5) + (2^2)(1/5) + (3^2)(1/15) = 9/5$.

4b. Methods #1 and #2 have nine terms that are $X^2 = (X_1 + X_2 + X_3)^2 = X_1X_1 + X_1X_2 + \cdots + X_3X_3$, so we get $\mathbb{E}(X^2) = \mathbb{E}((X_1 + X_2 + X_3)^2) = \mathbb{E}(X_1X_1) + \mathbb{E}(X_1X_2) + \cdots + \mathbb{E}(X_3X_3)$. By symmetry, this simplifies to $\mathbb{E}(X^2) = 3\mathbb{E}(X_1X_1) + 6\mathbb{E}(X_1X_2)$. We have $X_1X_1 = X_1$ (since X_1 is an indicator random variable), so we get $\mathbb{E}(X_1X_1) = \mathbb{E}(X_1) = 1/3$. To compute $\mathbb{E}(X_1X_2)$, note that there are 12 places where these two bear pairs can sit, each of which has probability $(2/6)(1/5)(2/4)(1/3) = 1/90$, and thus $\mathbb{E}(X_1X_2) = 12/90$. So altogether we get $\mathbb{E}(X^2) = (3)(1/3) + (6)(12/90) = 9/5$.

Method #3 has twenty five terms that are $X^2 = (X_1 + \cdots + X_5)^2 = X_1X_1 + X_1X_2 + \cdots + X_5X_5$, so we get $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_5)^2) = \mathbb{E}(X_1X_1) + \mathbb{E}(X_1X_2) + \cdots + \mathbb{E}(X_5X_5)$. There are 5 elements of the form $\mathbb{E}(X_1X_1)$ (all of which are $\mathbb{E}(X_1)$). There are 8 elements of the form $\mathbb{E}(X_jX_{j+1})$ (all of which are zero in this case). There are 12 elements of the form $\mathbb{E}(X_iX_j)$ in which i and j are (strictly) more than 1 apart (all of which are, by symmetry, the same). So this simplifies to $\mathbb{E}(X^2) = 5\mathbb{E}(X_1) + 8(0) + 12\mathbb{E}(X_1X_3)$. We know $\mathbb{E}(X_1) = 1/5$. We have $\mathbb{E}(X_1X_3) = (1/5)(1/3) = 1/15$. So altogether we get $\mathbb{E}(X^2) = (5)(1/5) + (8)(0) + (12)(1/15) = 9/5$.

4c. We get $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 9/5 - (1)^2 = 4/5$.